A1a **Vectors and Matrices: Example Sheet 1** Michaelmas 2014

A $*$ denotes a question, or part of a question, that should not be done at the expense of questions on later sheets. Starred questions are **not** *necessarily harder than unstarred questions.*

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1. Let S be the interior of the circle $|z-1-i|=1$. Show, by using suitable inequalities for $|z_1 \pm z_2|$, that if $z \in S$ then
$$\sqrt{5}-1 < |z-3| < \sqrt{5}+1.$$
Obtain the same result geometrically by considering the line containing the centre of the circle and the point 3.

2. Given $|z|=1$ and $\arg z = \theta$, find both algebraically and geometrically the modulus-argument forms of
$$\text{(i)}\quad 1+z, \qquad \text{(ii)}\quad 1-z\,.$$
Show that the locus of w as z varies with $|z|=1$, where w is given by
$$w^2 = \left(\frac{1-z}{1+z}\right),$$
is a pair of straight lines.

3. Use complex numbers to show that the medians of a triangle are concurrent.
Hint: represent the vertices of the triangle by complex numbers z_1, z_2 and z_3 (or 0, z_1 and z_2 if you prefer), then write down equations for two of the medians and find their intersection.

$*$4. Express
$$I = \frac{z^5-1}{z-1}$$
as a polynomial in z. By considering the complex fifth root of unity ω, obtain the four factors of I linear in z. Hence write I as the product of two real quadratic factors. By considering the term in z^2 in the identity so obtained for I, show that
$$4\cos\frac{\pi}{5}\sin\frac{\pi}{10} = 1\,.$$

5. Show the equation $\sin z = 2$ has infinitely many solutions.

6. (a) Let $z, a, b \in \mathbb{C}$ $(a \neq b)$ correspond to the points P, A, B of the Argand plane. Let C_λ be the locus of P defined by
$$PA/PB = \lambda\,,$$
where λ is a fixed real positive constant. Show that C_λ is a circle, unless $\lambda = 1$, and find its centre and radius. What if $\lambda = 1$?

$*$(b) For the case $a = -b = p$, $p \in \mathbb{R}$, and for each fixed $\mu \in \mathbb{R}$, show that the curve
$$S_\mu = \left\{ z \in \mathbb{C} : |z - i\mu| = \sqrt{p^2+\mu^2} \right\}$$
is a circle passing through A and B with its centre on the perpendicular bisector of AB.
Show that the circles C_λ and S_μ are orthogonal for all λ, μ.

7. Show by vector methods that the altitudes of a triangle are concurrent.
Hint: let the altitudes AD, BE of $\triangle ABC$ meet at H, and show that CH is perpendicular to AB.

8. Given that vectors $\mathbf{x}$ and $\mathbf{y}$ satisfy
$$\mathbf{x} + \mathbf{y}(\mathbf{x}\cdot\mathbf{y}) = \mathbf{a}\,,$$
for fixed vector $\mathbf{a}$, show that
$$(\mathbf{x}\cdot\mathbf{y})^2 = \frac{|\mathbf{a}|^2 - |\mathbf{x}|^2}{2+|\mathbf{y}|^2}\,.$$

Deduce using the Schwarz inequality (or otherwise) that

$$|\mathbf{x}|(1+|\mathbf{y}|^2) \geqslant |\mathbf{a}| \geqslant |\mathbf{x}| \,.$$

Explain the circumstances under which either of the inequalities can be replaced by equalities, and describe the relation between $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{a}$ in these circumstances.

9. (a) In $\triangle ABC$, let $\overrightarrow{AB}$, $\overrightarrow{BC}$ and $\overrightarrow{CA}$ be denoted by $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$. Show that

$$\mathbf{u}\times\mathbf{v} = \mathbf{v}\times\mathbf{w} = \mathbf{w}\times\mathbf{u}\,, \qquad (*)$$

and hence obtain the sine rule for $\triangle ABC$.

(b) Given *any* three vectors $\mathbf{p},\mathbf{q},\mathbf{r}$ such that

$$\mathbf{p}\times\mathbf{q} = \mathbf{q}\times\mathbf{r} = \mathbf{r}\times\mathbf{p}\,, \qquad (\dagger)$$

with $|\mathbf{p}\times\mathbf{q}| \neq 0$, show that

$$\mathbf{p}+\mathbf{q}+\mathbf{r} = \mathbf{0}\,.$$

10. (a) Using the identity $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}$, deduce that

$$\text{(i)}\quad (\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c}),$$
$$\text{(ii)}\quad \mathbf{a}\times(\mathbf{b}\times\mathbf{c}) + \mathbf{b}\times(\mathbf{c}\times\mathbf{a}) + \mathbf{c}\times(\mathbf{a}\times\mathbf{b}) = \mathbf{0}.$$

Relate the case $\mathbf{c}=\mathbf{a}$, $\mathbf{d}=\mathbf{b}$ of (i) to a well-known trigonometric identity.
Evaluate $(\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d})$ in two distinct ways and use the result to display explicitly a linear dependence relation amongst the four vectors $\mathbf{a},\mathbf{b},\mathbf{c}$ and $\mathbf{d}$.

(b) Given $[\mathbf{a},\mathbf{b},\mathbf{c}] \equiv \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$, show that

$$[\mathbf{a}\times\mathbf{b}, \mathbf{b}\times\mathbf{c}, \mathbf{c}\times\mathbf{a}] = [\mathbf{a},\mathbf{b},\mathbf{c}]^2.$$

$*$11. For $\phi,\theta\in\mathbb{R}$, let the vectors $\mathbf{e}_r$, $\mathbf{e}_\theta$ and $\mathbf{e}_\phi$ in $\mathbb{R}^3$ be defined in terms of the Cartesian basis $(\mathbf{i},\mathbf{j},\mathbf{k})$ by

$$\begin{aligned}
\mathbf{e}_r &= \cos\phi\sin\theta\,\mathbf{i} + \sin\phi\sin\theta\,\mathbf{j} + \cos\theta\,\mathbf{k}\,,\\
\mathbf{e}_\theta &= \cos\phi\cos\theta\,\mathbf{i} + \sin\phi\cos\theta\,\mathbf{j} - \sin\theta\,\mathbf{k}\,,\\
\mathbf{e}_\phi &= -\sin\phi\,\mathbf{i} + \cos\phi\,\mathbf{j}\,.
\end{aligned}$$

Show that $(\mathbf{e}_r,\mathbf{e}_\theta,\mathbf{e}_\phi)$ constitute an orthonormal right-handed basis. Discuss the significance of this [local] basis.

12. The set X contains the six real vectors

$$(1,1,0,0),\ (1,0,1,0),\ (1,0,0,1),\ (0,1,1,0),\ (0,1,0,1),\ (0,0,1,1).$$

Find two different subsets Y of X whose members are linearly independent, each of which yields a linearly dependent subset of X whenever any element $x\in X$ with $x\notin Y$ is adjoined to Y.

13. Let V be the set of all vectors $\mathbf{x}=(x_1,\ldots,x_n)$ in $\mathbb{R}^n$ $(n\geqslant 4)$ such that their components satisfy

$$x_i + x_{i+1} + x_{i+2} + x_{i+3} = 0 \quad \text{for} \quad i = 1,2,\ldots,n-3\,.$$

Find a basis for V.

14. Let $\mathbf{x}$ and $\mathbf{y}$ be non-zero vectors in $\mathbb{R}^n$ with scalar product denoted by $\mathbf{x}\cdot\mathbf{y}$. Prove that

$$(\mathbf{x}\cdot\mathbf{y})^2 \leqslant (\mathbf{x}\cdot\mathbf{x})(\mathbf{y}\cdot\mathbf{y})\,,$$

and prove also that $(\mathbf{x}\cdot\mathbf{y})^2 = (\mathbf{x}\cdot\mathbf{x})(\mathbf{y}\cdot\mathbf{y})$ if and only if $\mathbf{x}=\lambda\mathbf{y}$ for some scalar λ.

(a) By considering suitable vectors in $\mathbb{R}^3$, or otherwise, prove that the inequality

$$x^2+y^2+z^2 \geqslant yz+zx+xy$$

holds for any real numbers x, y and z.

(b) By considering suitable vectors in $\mathbb{R}^4$, or otherwise, show that only one choice of real numbers x, y and z satisfies

$$3(x^2+y^2+z^2+4) - 2(yz+zx+xy) - 4(x+y+z) = 0\,,$$

and find these numbers.

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A1b **Vectors and Matrices: Example Sheet 2** Michaelmas 2014

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1. In the following, the indices i, j, k, l take the values $1, 2, 3$, and the summation convention applies. In particular, $n_i n_i = 1$; i.e., n_i are the components of a unit vector $\mathbf{n}$.

   (a) Simplify the following expressions:
   $$\delta_{ij} a_j\,,\quad \delta_{ij}\delta_{jk}\,,\quad \delta_{ij}\delta_{ji}\,,\quad \delta_{ij} n_i n_j\,,\qquad \varepsilon_{ijk}\delta_{jk}\,,\qquad \varepsilon_{ijk}\varepsilon_{ijl}\,,\quad \varepsilon_{ijk}\varepsilon_{ikj}\,,\quad \varepsilon_{ijk}(\mathbf{a}\times\mathbf{b})_k.$$

   (b) Given that $A_{ij} = \varepsilon_{ijk} a_k$ (for all i, j), show that $2a_k = \varepsilon_{kij} A_{ij}$ (for all k).

   (c) Show that $\varepsilon_{ijk} s_{ij} = 0$ (for all k) if and only if $s_{ij} = s_{ji}$ (for all i, j).

   (d) Given that $N_{ij} = \delta_{ij} - \varepsilon_{ijk} n_k + n_i n_j$ and $M_{ij} = \delta_{ij} + \varepsilon_{ijk} n_k$, show that $N_{ij} M_{jk} = 2\delta_{ik}$.

2. Let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ and $\mathbf{d}$ be fixed vectors in $\mathbb{R}^3$. In each of cases (i) and (ii) find all vectors $\mathbf{r}$ such that
   $$\text{(i)}\quad \mathbf{r} + \mathbf{r}\times\mathbf{d} = \mathbf{c}\,, \qquad \text{(ii)}\quad \mathbf{r} + (\mathbf{r}\cdot\mathbf{a})\,\mathbf{b} = \mathbf{c}\,.$$
   In (ii) consider separately the $\mathbf{a}\cdot\mathbf{b} \neq -1$ and $\mathbf{a}\cdot\mathbf{b} = -1$ subcases.

   Hint: given $\mathbf{r}_0$ solving (ii) for $\mathbf{a}\cdot\mathbf{b} = -1$, show that $\mathbf{r}_0 + \lambda\mathbf{b}$ is another solution for an arbitrary scalar λ.

3. In $\mathbb{R}^3$ show that the straight line through the points $\mathbf{a}$ and $\mathbf{b}$ has equation
   $$\mathbf{r} \;=\; (1-\lambda)\mathbf{a} \;+\; \lambda\mathbf{b},$$
   and that the plane through the points $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ has the equation
   $$\mathbf{r} \;=\; (1-\mu-\nu)\mathbf{a} \;+\; \mu\mathbf{b} \;+\; \nu\mathbf{c},$$
   where λ, μ and ν are scalars. Obtain forms of these equations that do not involve λ, μ, ν.

4. (a) Let λ be a scalar, and let $\mathbf{m}, \mathbf{u}$ and $\mathbf{a}$ be fixed vectors in $\mathbb{R}^3$ such that $\mathbf{m}\cdot\mathbf{u} = 0$ and $\mathbf{a}\cdot\mathbf{u} \neq 0$. Show that the straight line $\mathbf{r}\times\mathbf{u} = \mathbf{m}$ meets the plane $\mathbf{r}\cdot\mathbf{a} = \lambda$ in the point
   $$\mathbf{r} \;=\; \frac{\mathbf{a}\times\mathbf{m} + \lambda\mathbf{u}}{\mathbf{a}\cdot\mathbf{u}}\,.$$
   Explain in detail the geometrical meaning of the condition $\mathbf{a}\cdot\mathbf{u} \neq 0$.

   (b) In $\mathbb{R}^3$ show that if $\mathbf{r}$ lies in the planes $\mathbf{r}\cdot\mathbf{a} = \lambda$ and $\mathbf{r}\cdot\mathbf{b} = \mu$, for fixed non-zero vectors $\mathbf{a}$ and $\mathbf{b}$, and scalars λ and μ, show that
   $$\mathbf{r}\times(\mathbf{a}\times\mathbf{b}) \;=\; \mu\mathbf{a} - \lambda\mathbf{b}\,. \qquad (*)$$
   Conversely, given $\mathbf{a}\times\mathbf{b} \neq \mathbf{0}$, show that $(*)$ implies both $\mathbf{r}\cdot\mathbf{a} = \lambda$ and $\mathbf{r}\cdot\mathbf{b} = \mu$. Hence deduce that the intersection of two non-parallel planes is a line. Comment on the case in which $\mathbf{a}\times\mathbf{b} = \mathbf{0}$.

5. Let $\mathbf{n}$ be a unit vector in $\mathbb{R}^3$. Identify the image and kernel (null space) of each of the following linear maps $\mathbb{R}^3 \to \mathbb{R}^3$:
   $$\text{(a)}\ \mathcal{T}: \mathbf{x} \mapsto \mathbf{x}' = \mathbf{x} - (\mathbf{x}\cdot\mathbf{n})\,\mathbf{n}\,, \qquad \text{(b)}\ \mathcal{Q}: \mathbf{x} \mapsto \mathbf{x}' = \mathbf{n}\times\mathbf{x}\,.$$
   Show that $\mathcal{T}^2 = \mathcal{T}$ and interpret the map $\mathcal{T}$ geometrically. Interpret the maps $\mathcal{Q}^2$ and $\mathcal{Q}^3 + \mathcal{Q}$, and show that $\mathcal{Q}^4 = \mathcal{T}$.

6. Give a geometrical description of the images and kernels of each of the linear maps of $\mathbb{R}^3$

   (a) $(x, y, z) \mapsto (x + 2y + z, x + 2y + z, 2x + 4y + 2z)$,

   (b) $(x, y, z) \mapsto (x + 2y + 3z, x - y + z, x + 5y + 5z)$.

7. A linear map $\mathcal{A}: \mathbb{R}^4 \to \mathbb{R}^4$ is defined by $\mathbf{x} \mapsto \mathsf{A}\mathbf{x}$ where
   $$\mathsf{A} \;=\; \begin{pmatrix} a & a & b & a \\ a & a & b & 0 \\ a & b & a & b \\ a & b & a & 0 \end{pmatrix}.$$
   Find the kernel and image of $\mathcal{A}$ for all real values of a and b.

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8. A linear map $\mathcal{S} : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by

$$\mathbf{x} \mapsto \mathbf{x}' = \mathbf{x} + \lambda(\mathbf{b}\cdot\mathbf{x})\,\mathbf{a}\,,$$

where λ is a scalar, and $\mathbf{a}$ and $\mathbf{b}$ are fixed, orthogonal unit vectors. By considering its effect on the vectors $\mathbf{a}$ and $\mathbf{b}$, show that $\mathcal{S}$ describes a shear in the direction of $\mathbf{a}$. Let $\mathsf{S}(\lambda, \mathbf{a}, \mathbf{b})$ be the matrix with entries S_{ij} such that $x'_i = S_{ij}x_j$. Obtain an expression for S_{ij} in terms of the components of $\mathbf{a}$ and $\mathbf{b}$ and hence find the matrix $\mathsf{S}(\lambda, \mathbf{a}, \mathbf{b})$. Evaluate its determinant‡, and hence deduce that $\mathcal{S}$ is an area-preserving map.

9. The linear map $\mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$\mathbf{x} \mapsto \mathbf{x}' = \cos\theta\,\mathbf{x} + (\mathbf{x}\cdot\mathbf{n})\,(1-\cos\theta)\;\mathbf{n} - \sin\theta\,(\mathbf{x}\times\mathbf{n}) \qquad (\dagger)$$

describes a rotation by angle θ in a positive sense about the unit vector $\mathbf{n}$. Verify this by considering the case of $\mathbf{n} = (0, 0, 1)$.

Show that (†) can be written in matrix form as

$$\mathbf{x} \mapsto \mathbf{x}' = \mathsf{R}(\mathbf{n}, \theta)\,\mathbf{x}$$

where $\mathsf{R}(\mathbf{n}, \theta)$ is a matrix with entries R_{ij} which you should find explicitly in terms of $\delta_{ij}, \varepsilon_{ijk}$, etc. Hence show that

$$R_{ii} = 2\cos\theta + 1 \quad , \quad \varepsilon_{ijk}R_{jk} = -2n_i\sin\theta.$$

Given that $\mathsf{R}(\mathbf{n}, \theta)$ is the matrix

$$\tfrac{1}{3}\begin{pmatrix} 2 & -1 & 2 \\ 2 & 2 & -1 \\ -1 & 2 & 2 \end{pmatrix},$$

determine θ and $\mathbf{n}$.

10. Give examples of 2×2 real matrices representing the following transformations in $\mathbb{R}^2$: (a) reflection, (b) dilatation (enlargement), (c) shear, and (d) rotation. Which of these types of transformation are always represented by a 2×2 matrix with determinant $+1$?

If maps $\mathcal{A}$ and $\mathcal{B}$ are both shears, will $\mathcal{AB}$ be the same as $\mathcal{BA}$ in general? Justify your answer.

11. Suppose that A and B are both Hermitian matrices. Show that $\mathsf{AB} + \mathsf{BA}$ is Hermitian. Also show that AB is Hermitian if and only if A and B commute.

∗12. Let $\mathsf{R}(\mathbf{n}, \theta)$ be the matrix defined by the linear map (†) of question 9, and let $\mathbf{i}, \mathbf{j}, \mathbf{k}$ be the standard mutually orthogonal unit vectors in $\mathbb{R}^3$.

(a) Show that the matrix $\mathsf{R}(\mathbf{i}, \frac{\pi}{2})\mathsf{R}(\mathbf{j}, \frac{\pi}{2})$ is orthogonal, has determinant one, and is not equal to the matrix $\mathsf{R}(\mathbf{j}, \frac{\pi}{2})\mathsf{R}(\mathbf{i}, \frac{\pi}{2})$.

(b) Reflection in a plane through the origin in $\mathbb{R}^3$, with unit normal $\mathbf{n}$, is a linear map such that

$$\mathbf{x} \mapsto \mathbf{x}' = \mathbf{x} - 2(\mathbf{x}\cdot\mathbf{n})\;\mathbf{n}.$$

In matrix notation $\mathbf{x}' = \mathsf{H}(\mathbf{n})\,\mathbf{x}$ for matrix $\mathsf{H}(\mathbf{n})$. Show by geometrical and algebraic means that the map $\mathbf{x} \mapsto \mathbf{x}' = -\mathsf{H}(\mathbf{n})\mathbf{x}$, describes a rotation of angle π about $\mathbf{n}$.

(c) A vector $\mathbf{x}$ has components (x, y, z) in a (Cartesian) coordinate system S. It has components (x', y', z') in a coordinate system S' obtained from S by anti-clockwise rotation through angle α about axis $\mathbf{k}$. Show, geometrically, that the components in coordinate system S' are related to those in S by

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \mathsf{R}(\mathbf{k}, -\alpha)\begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

(d) Given that

$$\mathbf{n}_\pm = \cos\left(\tfrac{1}{2}\theta\right)\,\mathbf{i} \;\pm\; \sin\left(\tfrac{1}{2}\theta\right)\,\mathbf{j}\,,$$

prove that

$$\mathsf{H}\,(\mathbf{i})\,\mathsf{H}\,(\mathbf{n}_-) = \mathsf{H}\,(\mathbf{n}_+)\,\mathsf{H}\,(\mathbf{i}) = \mathsf{R}\,(\mathbf{k}, \theta)\,,$$

and give diagrams to exhibit the geometrical meaning of this result.

‡ You may need to return to this question if determinants have not been covered yet.

A1c **Vectors and Matrices: Example Sheet 3** Michaelmas 2014

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1. Given that A is the real matrix
$$\begin{pmatrix} a & a^2 & bc \\ b & b^2 & ca \\ c & c^2 & ab \end{pmatrix},$$
show with the aid of row operations that
$$\det \mathsf{A} = (a-b)(b-c)(c-a)(ab+bc+ca).$$
[*Recall that the value of the determinant is unchanged if a linear combination of any two rows is added to the third row.*]

2. Show that
$$\begin{vmatrix} x & y & z \\ z & x & y \\ y & z & x \end{vmatrix} = x^3 + y^3 + z^3 - 3xyz \;\equiv\; \Delta.$$
Show, by row operations, that
$$x+y+z, \quad x+\omega y+\omega^2 z, \quad x+\omega^2 y+\omega z$$
are factors of Δ, where ω is a complex cube root of unity. Show, by considering the coefficients of x^3, that Δ is equal to the product of the three indicated factors.

3. If A is a $(2n+1)\times(2n+1)$ antisymmetric matrix ($n \in \mathbb{N}$), calculate $\det \mathsf{A}$.

4. Let D be the $n \times n$ matrix which has the entry p, $p \neq 1$, at each place on the main diagonal and unity in every other position. Show that $\det \mathsf{D} = (p+n-1)(p-1)^{n-1}$.

5. Identify the cofactors Δ_{ij} of a_{ij} in the matrix
$$\mathsf{A} = \{a_{ij}\} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 3 & -2 & 2 \end{pmatrix}.$$
Verify the identity $a_{ij}\Delta_{ik} = \delta_{jk} \det \mathsf{A}$, and hence construct the matrix A^{-1}.
Use your result to solve the equations
$$\begin{aligned} x+y+z &= 1\,, \\ x+2y+3z &= -5\,, \\ 3x-2y+2z &= 4\,. \end{aligned}$$
Verify that your answers for (x,y,z) do indeed satisfy the equations.

6. For each real value of t, determine whether or not there exist solutions to the simultaneous equations
$$\begin{aligned} x+y+z &= t \\ tx+2z &= 3 \\ 3x+ty+5z &= 7, \end{aligned}$$
exhibiting the most general form of such solutions when they exist.

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$*$7. Let A be a real 3×3 matrix, and let $\mathbf{d}$ be a 3 component column vector. Explain briefly how the general solution of the matrix equation $\mathsf{A}\mathbf{x} = \mathbf{d}$, where $\mathbf{x}$ is a 3 component column vector, depends on the kernel and image of the linear map $\mathbf{x} \mapsto \mathsf{A}\mathbf{x}$.

Consider the case

$$\mathsf{A} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & a & b \\ 1 & a^2 & b^2 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad \mathbf{d} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}.$$

Find the kernel and image of the corresponding map, noting the different possibilities according to different values of a and b.

For which values of a and b do the equations $\mathsf{A}\mathbf{x} = \mathbf{d}$ have (i) a unique solution, (ii) more than one solution, (iii) no solution? For each pair (a, b) satisfying (ii), give the solutions as the sum of a fixed solution and the general solution of the corresponding homogeneous equations.

8. Find the eigenvalues and eigenvectors of the matrix

$$\mathsf{A} = \begin{pmatrix} 1 & \alpha & 0 \\ \beta & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where neither of the complex constants α and β vanishes. Find the conditions for which (a) the eigenvalues are real, and (b) the eigenvectors are orthogonal. Hence show that both conditions are jointly satisfied if and only if A is Hermitian.

Recall both that the scalar product for two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{C}^3$ *is defined as*

$$\mathbf{u} \cdot \mathbf{v} = u_1^* v_1 + u_2^* v_2 + u_3^* v_3 \,,$$

where * *denotes a complex conjugate, and that* $\mathbf{u}$ *and* $\mathbf{v}$ *are said to be orthogonal if* $\mathbf{u} \cdot \mathbf{v} = 0$.

9. (a) Find a 3×3 real matrix with eigenvalues 1, i, $-i$. *Hint:* think geometrically.

   (b) Construct a 3×3 non-zero real matrix which has all three eigenvalues zero.

10. (a) Let A be a square matrix such that $\mathsf{A}^m = 0$ for some integer m. Show that every eigenvalue of A is zero.

    (b) Let A be a real 2×2 matrix which has non-zero non-real eigenvalues. Show that the non-diagonal elements of A are non-zero, but that the diagonal elements may be zero.

11. Let Q be a $(2n+1) \times (2n+1)$ orthogonal matrix ($n \in \mathbb{N}$) with $\det \mathsf{Q} = 1$. Show that Q has a unit eigenvalue. Give a geometric interpretation of your result for 3×3 matrices.

$*$12. Suppose that A is an $n \times n$ square matrix and that A^{-1} exists. Show that if A has characteristic equation $a_0 + a_1 t + \cdots + a_n t^n = 0$, then A^{-1} has characteristic equation

$$(-1)^n \det(\mathsf{A}^{-1})(a_n + a_{n-1} t + \cdots + a_0 t^n) = 0.$$

Hints. Take $n = 3$ in this question if you wish, but treat the general case if you can. It should be clear that λ is an eigenvalue of A if and only if $1/\lambda$ is an eigenvalue of A^{-1}, but this result says more than this (about multiplicities of eigenvalues). You should use properties of the determinant to solve this problem, for example, $\det(\mathsf{A})\det(\mathsf{B}) = \det(\mathsf{AB})$. You should also state explicitly why we do not need to worry about zero eigenvalues.

13. For each of the three matrices below,

    (a) compute their eigenvalues (as often happens in exercises and seldom in real life each eigenvalue is a small integer);

    (b) for each real eigenvalue λ compute the dimension of the eigenspace $\{\mathbf{x} \in \mathbb{R}^3 : \mathsf{A}\mathbf{x} = \lambda\mathbf{x}\}$;

    (c) determine whether or not the matrix is diagonalizable as a map of $\mathbb{R}^3$ into itself.

$$\begin{pmatrix} 5 & -3 & 2 \\ 6 & -4 & 4 \\ 4 & -4 & 5 \end{pmatrix}, \quad \begin{pmatrix} 1 & -3 & 4 \\ 4 & -7 & 8 \\ 6 & -7 & 7 \end{pmatrix}, \quad \begin{pmatrix} 7 & -12 & 6 \\ 10 & -19 & 10 \\ 12 & -24 & 13 \end{pmatrix}.$$

A1d **Vectors and Matrices: Example Sheet 4** Michaelmas 2014

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1. A matrix A is said to be *upper triangular* if $A_{ij} = 0$ if $i > j$, i.e. if

$$\mathsf{A} = \begin{pmatrix} A_{11} & A_{12} & \dots & A_{1\,n} \\ 0 & A_{22} & \ddots & A_{2\,n} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & A_{nn} \end{pmatrix}.$$

Show that the eigenvalues are $\lambda_i = A_{ii}$ ($i = 1, \dots, n$, and obviously no sum).

2. Let $\{\mathbf{e}_1, \dots, \mathbf{e}_m\}$ and $\{\mathbf{f}_1, \dots, \mathbf{f}_n\}$ be bases for $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively, and let $\mathcal{A}$ be a linear mapping from $\mathbb{R}^m$ to $\mathbb{R}^n$. Explain how to represent $\mathcal{A}$ by a matrix relative to the given bases.

(a) Taking $m = 2$, $n = 3$ and $\mathcal{A}$ as the linear mapping for which

$$\mathcal{A}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}2\\1\\5\end{pmatrix}, \quad \mathcal{A}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}7\\0\\3\end{pmatrix},$$

where components are with respect to the standard bases for $\mathbb{R}^2$ and $\mathbb{R}^3$, find the matrix of $\mathcal{A}$ with respect to the bases

$$\mathbf{e}_1 = \begin{pmatrix}1\\-1\end{pmatrix}, \quad \mathbf{e}_2 = \begin{pmatrix}-3\\2\end{pmatrix}; \qquad \mathbf{f}_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}, \quad \mathbf{f}_2 = \begin{pmatrix}0\\1\\0\end{pmatrix}, \quad \mathbf{f}_3 = \begin{pmatrix}0\\0\\1\end{pmatrix}.$$

(b) The mapping $\mathcal{A}$ of $\mathbb{R}^3$ to itself is a reflection in the plane $x_1 \sin\theta = x_2 \cos\theta$. Find the matrix A of $\mathcal{A}$ with respect to any basis of your choice, which should be specified.

3. The linear map $\mathcal{A} : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by

$$\begin{pmatrix}x\\y\end{pmatrix} \longmapsto \begin{pmatrix}x'\\y'\end{pmatrix} = \mathsf{A}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}-5x+9y\\-4x+7y\end{pmatrix}.$$

Find the matrix B of the map $\mathcal{A}$ relative to the basis

$$\left\{\begin{pmatrix}3\\2\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}\right\},$$

and interpret the map geometrically. Hence show that, for each positive integer n,

$$\mathsf{B}^n - \mathsf{I} = n(\mathsf{B} - \mathsf{I}),$$

where $\mathsf{I} = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$. Hence evaluate A^n. Verify that $\det(\mathsf{A}^n) = (\det \mathsf{A})^n$.

$*$4. Show that similar matrices have the same rank.

5. Show that the matrix

$$\mathsf{A} = \begin{pmatrix}0 & 1 & 0\\-4 & 4 & 0\\-2 & 1 & 2\end{pmatrix}$$

has characteristic equation $(t-2)^3 = 0$. Explain (without doing any further calculations) why A is not diagonalisable.

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6. (a) Find a, b and c such that the matrix

$$\begin{pmatrix} 1/3 & 0 & a \\ 2/3 & 1/\sqrt{2} & b \\ 2/3 & -1/\sqrt{2} & c \end{pmatrix}$$

is orthogonal. Does this condition determine a, b and c uniquely?

(b) Let

$$\mathsf{A} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } \mathsf{P} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

Do you expect PAP^{-1} to be symmetric? Compute PAP^{-1}. Were you right?

∗7. (a) Show that the Cayley-Hamilton theorem is true for all 2×2 matrices.

(b) Let

$$\mathsf{A} = \begin{pmatrix} 3 & 4 \\ -1 & -1 \end{pmatrix}.$$

Find the characteristic equation for A. Verify that $\mathsf{A}^2 = 2\mathsf{A} - \mathsf{I}$. Is A diagonalisable ?

Show by induction that A^n lies in the two-dimensional subspace (of the space of 2×2 real matrices) spanned by A and I, i.e. show that there exists real numbers α_n and β_n such that

$$\mathsf{A}^n = \alpha_n \mathsf{A} + \beta_n \mathsf{I}\,.$$

Find a recurrence relation (i.e. a difference equation) for α_n and β_n, and hence find an explicit formula for A^n.

8. Determine the eigenvalues and eigenvectors of the symmetric matrix

$$\mathsf{A} = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}.$$

Use an identity of the form $\mathsf{P}^{\mathrm{T}}\mathsf{AP} = \mathsf{D}$, where D is a diagonal matrix, to find A^{-1}.

∗9. Show that the eigenvalues of a unitary matrix have unit modulus. Show that if a unitary matrix has distinct eigenvalues then the eigenvectors are orthogonal.

10. A skew-Hermitian matrix, W, is one such that $\mathsf{W}^\dagger = -\mathsf{W}$. What can be said about the eigenvalues of a skew-Hermitian matrix? (*Hint: consider* $\mathsf{H} = i\mathsf{W}$)?

If S is real symmetric and T is real skew-symmetric, show that $\mathsf{T} \pm i\mathsf{S}$ is skew-Hermitian. State a property of the eigenvalues of $\mathsf{T} + i\mathsf{S}$ and hence, or otherwise, show that

$$\det(\mathsf{T} + i\mathsf{S} - \mathsf{I}) \neq 0\,.$$

Show that the matrix

$$\mathsf{U} = (\mathsf{I} + \mathsf{T} + i\mathsf{S})(\mathsf{I} - \mathsf{T} - i\mathsf{S})^{-1}$$

is unitary. For

$$\mathsf{S} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \qquad \mathsf{T} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

show that the eigenvalues of U are $\pm(1-i)/\sqrt{2}$.

∗11. *This is a continuation of question 8 on Example Sheet 2.*

As in question 8 on Example Sheet 2 consider the linear map $\mathcal{S} : \mathbb{R}^2 \to \mathbb{R}^2$

$$\mathbf{x} \mapsto \mathbf{x}' = \mathbf{x} + \lambda(\mathbf{b} \cdot \mathbf{x})\,\mathbf{a} \qquad (*)$$

where λ is a real scalar, $\mathbf{a}$ and $\mathbf{b}$ are fixed orthogonal unit vectors. Let $\mathsf{S}(\lambda, \mathbf{a}, \mathbf{b})$ be the matrix with elements S_{ij} such that $x'_i = S_{ij} x_j$. Give diagrams illustrating the shears

$$\mathsf{S}_1 = \mathsf{S}(\lambda, \mathbf{i}, \mathbf{j})\,, \quad \mathsf{S}_2 = \mathsf{S}(\lambda, \mathbf{j}, -\mathbf{i}).$$

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Show that S_1 and S_2 are related by a similarity transformation

$$\mathsf{S}_2 = \mathsf{R}^{-1}\mathsf{S}_1\mathsf{R}\,, \quad \mathsf{R} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Now let $\mathcal{S}$ be the map defined by $(*)$ but from $\mathbb{R}^3$ to $\mathbb{R}^3$, and let $\mathbf{i}, \mathbf{j}, \mathbf{k}$ be unit vectors along the three perpendicular axes. Find the matrix S in each of the cases

$$\text{(i)} \quad \mathbf{a} = \mathbf{i}\,,\ \mathbf{b} = \mathbf{j}\,, \qquad \text{(ii)} \quad \mathbf{a} = \mathbf{j}\,,\ \mathbf{b} = \mathbf{k}\,, \qquad \text{(iii)} \quad \mathbf{a} = \mathbf{k}\,,\ \mathbf{b} = \mathbf{i}\,,$$

and interpret the corresponding simple shears. Show that any matrix of the form

$$\begin{pmatrix} 1 & \lambda & \mu \\ 0 & 1 & \nu \\ 0 & 0 & 1 \end{pmatrix}$$

can be displayed (not necessarily uniquely) as the product of matrices of simple shears.

$*$12. Diagonalize the quadratic form

$$\mathcal{F} = (a\cos^2\theta + b\sin^2\theta)x^2 + 2(a-b)(\sin\theta\cos\theta)xy + (a\sin^2\theta + b\cos^2\theta)y^2,$$

and identify the principal axes.

13. Find all eigenvalues, and an orthonormal set of eigenvectors, of the matrices

$$\mathsf{A} = \begin{pmatrix} 5 & 0 & \sqrt{3} \\ 0 & 3 & 0 \\ \sqrt{3} & 0 & 3 \end{pmatrix} \quad \text{and} \quad \mathsf{B} = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2. \end{pmatrix}$$

Hence sketch the surfaces

$$5x^2 + 3y^2 + 3z^2 + 2\sqrt{3}xz = 1 \quad \text{and} \quad x^2 + y^2 + z^2 - xy - yz - zx = 1\,.$$

14. Let Σ be the surface in $\mathbb{R}^3$ given by

$$2x^2 + 2xy + 4yz + z^2 = 1.$$

By writing this equation as

$$\mathbf{x}^{\mathrm{T}}\mathsf{A}\mathbf{x} = 1,$$

with A a real symmetric matrix, show that there is an orthonormal basis such that, if we use coordinates (u, v, w) with respect to this new basis, Σ takes the form

$$\lambda u^2 + \mu v^2 + \nu w^2 = 1.$$

Find λ, μ and ν and hence find the minimum distance between the origin and Σ. *Hint: it is **not** necessary to find the basis explicitly.*

15. (i) Explain what is meant by saying that a 2×2 real matrix,

$$\mathsf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

preserves the scalar product on $\mathbb{R}^2$ with respect to

$$\text{(a) the Euclidean metric, } \mathsf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \text{or (b) the Minkowski metric, } \mathsf{J} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(ii) Using a single real parameter together with a choice of sign (± 1), give and justify a description of all matrices, A, that preserve the scalar product with respect to the Euclidean metric. Show that these matrices form a group.

(iii) Using a single real parameter together with a choice of sign (± 1), give and justify a description of all matrices, A with $a > 0$, that preserve the scalar product with respect to the Minkowski metric. Show that these matrices form a group.

(iv) What is the intersection of the above two groups?

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Revision Questions

16. Show that a rotation about the z axis through an angle θ corresponds to the matrix

$$\mathsf{R} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Write down a real eigenvector of R and give the corresponding eigenvalue. In the case of a matrix corresponding to a general rotation, how can one find the axis of rotation?

A rotation through 45° about the x-axis is followed by a similar one about the z-axis. Show that the rotation corresponding to their combined effect has its axis inclined at equal angles

$$\cos^{-1} \frac{1}{\sqrt{(5-2\sqrt{2})}}$$

to the x and z axes.

17. Show that the eigenvalues of a real orthogonal matrix have unit modulus and that if λ is an eigenvalue so is λ^*. Hence argue that the eigenvalues of a 3×3 real orthogonal matrix Q must be a selection from

$$+1, \quad -1 \quad \text{and} \quad \mathrm{e}^{i\alpha} \ \& \ \mathrm{e}^{-i\alpha}.$$

Verify that $\det \mathsf{Q} = \pm 1$. What is the effect of Q on vectors orthogonal to an eigenvector with eigenvalue ± 1?

$*$18. *This is another way of proving* $\det \mathsf{AB} = \det \mathsf{A} \det \mathsf{B}$. *You may wish to stick to the case* $n = 3$.

If $1 \leqslant r, s \leqslant n$, $r \neq s$ and λ is real, let $E(\lambda, r, s)$ be an $n \times n$ matrix with (i,j) entry $\delta_{ij} + \lambda \delta_{ir}\delta_{js}$. If $1 \leqslant r \leqslant n$ and μ is real, let $F(\mu, r)$ be an $n \times n$ matrix with (i,j) entry $\delta_{ij} + (\mu - 1)\delta_{ir}\delta_{jr}$.

(i) Give a simple geometric interpretation of the linear maps from $\mathbb{R}^n$ to $\mathbb{R}^n$ associated with $\mathsf{E}(\lambda, r, s)$ and $\mathsf{F}(\mu, r)$.

(ii) Give a simple account of the effect of pre-multiplying an $n \times m$ matrix by $\mathsf{E}(\lambda, r, s)$ and by $\mathsf{F}(\mu, r)$. What is the effect of post-multiplying an $m \times n$ matrix?

(iii) If A is an $n \times n$ matrix, find $\det(\mathsf{E}(\lambda, r, s)\mathsf{A})$ and $\det(\mathsf{F}(\mu, r)\mathsf{A})$ in terms of $\det \mathsf{A}$.

(iv) Show that every $n \times n$ matrix is the product of matrices of the form $\mathsf{E}(\lambda, r, s)$ and $\mathsf{F}(\mu, r)$ and a diagonal matrix with entries 0 or 1.

(v) Use (iii) and (iv) to show that, if A and B are $n \times n$ matrices, then $\det \mathsf{A} \det \mathsf{B} = \det \mathsf{AB}$.

$*$19. *The object of this exercise is to show why finding eigenvalues of a large matrix is not just a matter of finding a large fast computer.*

Consider the $n \times n$ complex matrix $\mathsf{A} = \{A_{ij}\}$ given by

$$\begin{aligned} A_{j\,j+1} &= 1 \qquad \text{for } 1 \leqslant j \leqslant n-1 \\ A_{n1} &= \kappa^n \\ A_{ij} &= 0 \qquad \text{otherwise,} \end{aligned}$$

where $\kappa \in \mathbb{C}$ is non-zero. Thus, when $n = 2$ and $n = 3$, we get the matrices

$$\begin{pmatrix} 0 & 1 \\ \kappa^2 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \kappa^3 & 0 & 0 \end{pmatrix}.$$

(i) Find the eigenvalues and associated eigenvectors of A for $n = 2$ and $n = 3$.

(ii) By guessing and then verifying your answers, or otherwise, find the eigenvalues and associated eigenvectors of A for for all $n \geqslant 2$.

(iii) Suppose that your computer works to 15 decimal places and that $n = 100$. You decide to find the eigenvalues of A in the cases $\kappa = 2^{-1}$ and $\kappa = 3^{-1}$. Explain why at least one (and more probably) both attempts will deliver answers which bear no relation to the true answers.