Part IA — Vectors and Matrices
Definitions

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Complex numbers
Review of complex numbers, including complex conjugate, inverse, modulus, argument and Argand diagram. Informal treatment of complex logarithm, n-th roots and complex powers. de Moivre's theorem. [2]

Vectors
Review of elementary algebra of vectors in $\mathbb{R}^3$, including scalar product. Brief discussion of vectors in $\mathbb{R}^n$ and $\mathbb{C}^n$; scalar product and the Cauchy-Schwarz inequality. Concepts of linear span, linear independence, subspaces, basis and dimension.

Suffix notation: including summation convention, δ_{ij} and ε_{ijk}. Vector product and triple product: definition and geometrical interpretation. Solution of linear vector equations. Applications of vectors to geometry, including equations of lines, planes and spheres. [5]

Matrices
Elementary algebra of 3×3 matrices, including determinants. Extension to $n \times n$ complex matrices. Trace, determinant, non-singular matrices and inverses. Matrices as linear transformations; examples of geometrical actions including rotations, reflections, dilations, shears; kernel and image. [4]

Simultaneous linear equations: matrix formulation; existence and uniqueness of solutions, geometric interpretation; Gaussian elimination. [3]

Symmetric, anti-symmetric, orthogonal, hermitian and unitary matrices. Decomposition of a general matrix into isotropic, symmetric trace-free and antisymmetric parts. [1]

Eigenvalues and Eigenvectors
Eigenvalues and eigenvectors; geometric significance. [2]

Proof that eigenvalues of hermitian matrix are real, and that distinct eigenvalues give an orthogonal basis of eigenvectors. The effect of a general change of basis (similarity transformations). Diagonalization of general matrices: sufficient conditions; examples of matrices that cannot be diagonalized. Canonical forms for 2×2 matrices. [5]

Discussion of quadratic forms, including change of basis. Classification of conics, cartesian and polar forms. [1]

Rotation matrices and Lorentz transformations as transformation groups. [1]

Contents

0 Introduction

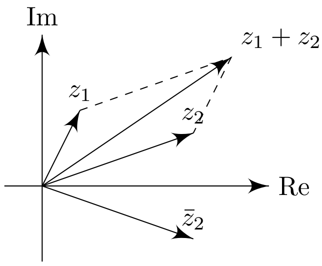
1 Complex numbers

1.1 Basic properties

Definition (Complex number)**.** A *complex number* is a number $z \in \mathbb{C}$ of the form $z = a + ib$ with $a, b \in \mathbb{R}$, where $i^2 = -1$. We write $a = \mathrm{Re}(z)$ and $b = \mathrm{Im}(z)$.

Definition (Complex conjugate)**.** The *complex conjugate* of $z = a + ib$ is $a - ib$. It is written as $\bar{z}$ or z^*.

Definition (Argand diagram)**.** An *Argand diagram* is a diagram in which a complex number $z = x + iy$ is represented by a vector $\mathbf{p} = \begin{pmatrix} x \\ y \end{pmatrix}$. Addition of vectors corresponds to vector addition and $\bar{z}$ is the reflection of z in the x-axis.



Definition (Modulus and argument of complex number)**.** The *modulus* of $z = x + iy$ is $r = |z| = \sqrt{x^2 + y^2}$. The *argument* is $\theta = \arg z = \tan^{-1}(y/x)$. The modulus is the length of the vector in the Argand diagram, and the argument is the angle between z and the real axis. We have

$$z = r(\cos\theta + i\sin\theta)$$

Clearly the pair (r, θ) uniquely describes a complex number z, but each complex number $z \in \mathbb{C}$ can be described by many different θ since $\sin(2\pi + \theta) = \sin\theta$ and $\cos(2\pi + \theta) = \cos\theta$. Often we take the *principle value* $\theta \in (-\pi, \pi]$.

1.2 Complex exponential function

Definition (Exponential function)**.** The *exponential function* is defined as

$$\exp(z) = e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

Definition (Sine and cosine functions)**.** Define, for all $z \in \mathbb{C}$,

$$\begin{aligned} \sin z &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1} &&= z - \frac{1}{3!}z^3 + \frac{1}{5!}z^5 + \cdots \\ \cos z &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n} &&= 1 - \frac{1}{2!}z^2 + \frac{1}{4!}z^4 + \cdots \end{aligned}$$

1.3 Roots of unity

Definition (Roots of unity)**.** The nth *roots of unity* are the roots to the equation $z^n = 1$ for $n \in \mathbb{N}$. Since this is a polynomial of order n, there are n roots of unity. In fact, the nth roots of unity are $\exp\left(2\pi i \frac{k}{n}\right)$ for $k = 0, 1, 2, 3 \cdots n - 1$.

1.4 Complex logarithm and power

Definition (Complex logarithm)**.** The *complex logarithm* $w = \log z$ is a solution to $e^\omega = z$, i.e. $\omega = \log z$. Writing $z = re^{i\theta}$, we have $\log z = \log(re^{i\theta}) = \log r + i\theta$. This can be multi-valued for different values of θ and, as above, we should select the θ that satisfies $-\pi < \theta \leq \pi$.

Definition (Complex power)**.** The *complex power* z^α for $z, \alpha \in \mathbb{C}$ is defined as $z^\alpha = e^{\alpha \log z}$. This, again, can be multi-valued, as $z^\alpha = e^{\alpha \log |z|} e^{i\alpha\theta} e^{2in\pi\alpha}$ (there are finitely many values if $\alpha \in \mathbb{Q}$, infinitely many otherwise). Nevertheless, we make z^α single-valued by insisting $-\pi < \theta \leq \pi$.

1.5 De Moivre's theorem

1.6 Lines and circles in $\mathbb{C}$

2 Vectors

2.1 Definition and basic properties

Definition (Vector)**.** A *vector space* over $\mathbb{R}$ or $\mathbb{C}$ is a collection of vectors $\mathbf{v} \in V$, together with two operations: addition of two vectors and multiplication of a vector with a scalar (i.e. a number from $\mathbb{R}$ or $\mathbb{C}$, respectively).

Vector addition has to satisfy the following axioms:

(i) $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$ (commutativity)

(ii) $(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$ (associativity)

(iii) There is a vector $\mathbf{0}$ such that $\mathbf{a} + \mathbf{0} = \mathbf{a}$. (identity)

(iv) For all vectors $\mathbf{a}$, there is a vector $(-\mathbf{a})$ such that $\mathbf{a} + (-\mathbf{a}) = \mathbf{0}$ (inverse)

Scalar multiplication has to satisfy the following axioms:

(i) $\lambda(\mathbf{a} + \mathbf{b}) = \lambda\mathbf{a} + \lambda\mathbf{b}$.

(ii) $(\lambda + \mu)\mathbf{a} = \lambda\mathbf{a} + \mu\mathbf{a}$.

(iii) $\lambda(\mu\mathbf{a}) = (\lambda\mu)\mathbf{a}$.

(iv) $1\mathbf{a} = \mathbf{a}$.

Definition (Unit vector)**.** A *unit vector* is a vector with length 1. We write a unit vector as $\hat{\mathbf{v}}$.

2.2 Scalar product

2.2.1 Geometric picture ($\mathbb{R}^2$ and $\mathbb{R}^3$ only)

Definition (Scalar/dot product)**.** $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$, where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$. It satisfies the following properties:

(i) $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$

(ii) $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2 \geq 0$

(iii) $\mathbf{a} \cdot \mathbf{a} = 0$ iff $\mathbf{a} = \mathbf{0}$

(iv) If $\mathbf{a} \cdot \mathbf{b} = 0$ and $\mathbf{a}, \mathbf{b} \neq \mathbf{0}$, then $\mathbf{a}$ and $\mathbf{b}$ are perpendicular.

2.2.2 General algebraic definition

Definition (Inner/scalar product)**.** In a real vector space V, an *inner product* or *scalar product* is a map $V \times V \to \mathbb{R}$ that satisfies the following axioms. It is written as $\mathbf{x} \cdot \mathbf{y}$ or $\langle \mathbf{x} \mid \mathbf{y} \rangle$.

(i) $\mathbf{x} \cdot \mathbf{y} = \mathbf{y} \cdot \mathbf{x}$ (symmetry)

(ii) $\mathbf{x} \cdot (\lambda\mathbf{y} + \mu\mathbf{z}) = \lambda\mathbf{x} \cdot \mathbf{y} + \mu\mathbf{x} \cdot \mathbf{z}$ (linearity in 2nd argument)

(iii) $\mathbf{x} \cdot \mathbf{x} \geq 0$ with equality iff $\mathbf{x} = \mathbf{0}$ (positive definite)

Definition. The *norm* of a vector, written as $|\mathbf{a}|$ or $\|\mathbf{a}\|$, is defined as

$$|\mathbf{a}| = \sqrt{\mathbf{a} \cdot \mathbf{a}}.$$

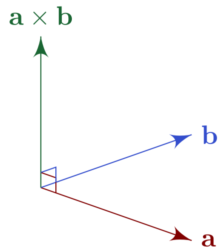
2.3 Cauchy-Schwarz inequality

2.4 Vector product

Definition (Vector/cross product). Consider $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$. Define the *vector product*

$$\mathbf{a} \times \mathbf{b} = |\mathbf{a}||\mathbf{b}| \sin\theta \hat{\mathbf{n}},$$

where $\hat{\mathbf{n}}$ is a unit vector perpendicular to both $\mathbf{a}$ and $\mathbf{b}$. Since there are two (opposite) unit vectors that are perpendicular to both of them, we pick $\hat{\mathbf{n}}$ to be the one that is perpendicular to $\mathbf{a}, \mathbf{b}$ in a *right-handed* sense.



The vector product satisfies the following properties:

(i) $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$.

(ii) $\mathbf{a} \times \mathbf{a} = \mathbf{0}$.

(iii) $\mathbf{a} \times \mathbf{b} = \mathbf{0} \Rightarrow \mathbf{a} = \lambda \mathbf{b}$ for some $\lambda \in \mathbb{R}$ (or $\mathbf{b} = \mathbf{0}$).

(iv) $\mathbf{a} \times (\lambda \mathbf{b}) = \lambda(\mathbf{a} \times \mathbf{b})$.

(v) $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$.

2.5 Scalar triple product

Definition (Scalar triple product). The *scalar triple product* is defined as

$$[\mathbf{a}, \mathbf{b}, \mathbf{c}] = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}).$$

2.6 Spanning sets and bases

2.6.1 2D space

Definition (Spanning set). A set of vectors $\{\mathbf{a}, \mathbf{b}\}$ *spans* $\mathbb{R}^2$ if for all vectors $\mathbf{r} \in \mathbb{R}^2$, there exist some $\lambda, \mu \in \mathbb{R}$ such that $\mathbf{r} = \lambda \mathbf{a} + \mu \mathbf{b}$.

Definition (Linearly independent vectors in $\mathbb{R}^2$). Two vectors $\mathbf{a}$ and $\mathbf{b}$ are *linearly independent* if for $\alpha, \beta \in \mathbb{R}$, $\alpha\mathbf{a} + \beta\mathbf{b} = \mathbf{0}$ iff $\alpha = \beta = 0$. In $\mathbb{R}^2$, $\mathbf{a}$ and $\mathbf{b}$ are linearly independent if $\mathbf{a} \times \mathbf{b} \neq \mathbf{0}$.

Definition (Basis of $\mathbb{R}^2$). A set of vectors is a *basis* of $\mathbb{R}^2$ if it spans $\mathbb{R}^2$ and are linearly independent.

2.6.2 3D space

2.6.3 $\mathbb{R}^n$ space

Definition (Linearly independent vectors)**.** A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \cdots \mathbf{v}_m\}$ is *linearly independent* if

$$\sum_{i=1}^{m} \lambda_i \mathbf{v}_i = \mathbf{0} \Rightarrow (\forall i)\, \lambda_i = 0.$$

Definition (Spanning set)**.** A set of vectors $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3 \cdots \mathbf{u}_m\} \subseteq \mathbb{R}^n$ is a *spanning set* of $\mathbb{R}^n$ if

$$(\forall \mathbf{x} \in \mathbb{R}^n)(\exists \lambda_i) \sum_{i=1}^{m} \lambda_i \mathbf{u}_i = \mathbf{x}$$

Definition (Basis vectors)**.** A *basis* of $\mathbb{R}^n$ is a linearly independent spanning set. The standard basis of $\mathbb{R}^n$ is $\mathbf{e}_1 = (1, 0, 0, \cdots 0), \mathbf{e}_2 = (0, 1, 0, \cdots 0), \cdots \mathbf{e}_n = (0, 0, 0, \cdots, 1)$.

Definition (Orthonormal basis)**.** A basis $\{\mathbf{e}_i\}$ is *orthonormal* if $\mathbf{e}_i \cdot \mathbf{e}_j = 0$ if $i \neq j$ and $\mathbf{e}_i \cdot \mathbf{e}_i = 1$ for all i, j.

Using the Kronecker Delta symbol, which we will define later, we can write this condition as $\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$.

Definition (Dimension of vector space)**.** The *dimension* of a vector space is the number of vectors in its basis. (Exercise: show that this is well-defined)

Definition (Scalar product)**.** The *scalar product* of $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ is defined as $\mathbf{x} \cdot \mathbf{y} = \sum x_i y_i$.

2.6.4 $\mathbb{C}^n$ space

Definition ($\mathbb{C}^n$)**.** $\mathbb{C}^n = \{(z_1, z_2, \cdots, z_n) : z_i \in \mathbb{C}\}$. It has the same standard basis as $\mathbb{R}^n$ but the scalar product is defined differently. For $\mathbf{u}, \mathbf{v} \in \mathbb{C}^n$, $\mathbf{u} \cdot \mathbf{v} = \sum u_i^* v_i$. The scalar product has the following properties:

(i) $\mathbf{u} \cdot \mathbf{v} = (\mathbf{v} \cdot \mathbf{u})^*$

(ii) $\mathbf{u} \cdot (\lambda \mathbf{v} + \mu \mathbf{w}) = \lambda(\mathbf{u} \cdot \mathbf{v}) + \mu(\mathbf{u} \cdot \mathbf{w})$

(iii) $\mathbf{u} \cdot \mathbf{u} \geq 0$ and $\mathbf{u} \cdot \mathbf{u} = 0$ iff $\mathbf{u} = \mathbf{0}$

2.7 Vector subspaces

Definition (Vector subspace)**.** A *vector subspace* of a vector space V is a subset of V that is also a vector space under the same operations. Both V and $\{\mathbf{0}\}$ are subspaces of V. All others are proper subspaces.

A useful criterion is that a subset $U \subseteq V$ is a subspace iff

(i) $\mathbf{x}, \mathbf{y} \in U \Rightarrow (\mathbf{x} + \mathbf{y}) \in U$.

(ii) $\mathbf{x} \in U \Rightarrow \lambda \mathbf{x} \in U$ for all scalars λ.

(iii) $\mathbf{0} \in U$.

This can be more concisely written as "U is non-empty and for all $\mathbf{x}, \mathbf{y} \in U$, $(\lambda \mathbf{x} + \mu \mathbf{y}) \in U$".

2.8 Suffix notation

Notation (Einstein's summation convention)**.** Consider a sum $\mathbf{x}\cdot\mathbf{y} = \sum x_i y_i$. The *summation convention* says that we can drop the $\sum$ symbol and simply write $\mathbf{x}\cdot\mathbf{y} = x_i y_i$. If suffixes are repeated once, summation is understood.

Note that i is a dummy suffix and doesn't matter what it's called, i.e. $x_i y_i = x_j y_j = x_k y_k$ etc.

The rules of this convention are:

(i) Suffix appears once in a term: free suffix

(ii) Suffix appears twice in a term: dummy suffix and is summed over

(iii) Suffix appears three times or more: WRONG!

Definition (Kronecker delta)**.**

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}.$$

We have

$$\begin{pmatrix} \delta_{11} & \delta_{12} & \delta_{13} \\ \delta_{21} & \delta_{22} & \delta_{23} \\ \delta_{31} & \delta_{32} & \delta_{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbf{I}.$$

So the Kronecker delta represents an identity matrix.

Definition (Alternating symbol ε_{ijk})**.** Consider rearrangements of $1, 2, 3$. We can divide them into even and odd permutations. Even permutations include $(1, 2, 3)$, $(2, 3, 1)$ and $(3, 1, 2)$. These are permutations obtained by performing two (or no) swaps of the elements of $(1, 2, 3)$. (Alternatively, it is any "rotation" of $(1, 2, 3)$)

The odd permutations are $(2, 1, 3)$, $(1, 3, 2)$ and $(3, 2, 1)$. They are the permutations obtained by one swap only.

Define

$$\varepsilon_{ijk} = \begin{cases} +1 & ijk \text{ is even permutation} \\ -1 & ijk \text{ is odd permutation} \\ 0 & \text{otherwise (i.e. repeated suffices)} \end{cases}$$

ε_{ijk} has 3 free suffices.

We have $\varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} = +1$ and $\varepsilon_{213} = \varepsilon_{132} = \varepsilon_{321} = -1$. $\varepsilon_{112} = \varepsilon_{111} = \cdots = 0$.

2.9 Geometry

2.9.1 Lines

2.9.2 Plane

2.10 Vector equations

3 Linear maps

3.1 Examples

3.1.1 Rotation in $\mathbb{R}^3$

3.1.2 Reflection in $\mathbb{R}^3$

3.2 Linear Maps

Definition (Domain, codomain and image of map)**.** Consider sets A and B and mapping $T : A \to B$ such that each $x \in A$ is mapped into a unique $x' = T(x) \in B$. A is the *domain* of T and B is the *co-domain* of T. Typically, we have $T : \mathbb{R}^n \to \mathbb{R}^m$ or $T : \mathbb{C}^n \to \mathbb{C}^m$.

Definition (Linear map)**.** Let V, W be real (or complex) vector spaces, and $T : V \to W$. Then T is a *linear map* if

(i) $T(\mathbf{a} + \mathbf{b}) = T(\mathbf{a}) + T(\mathbf{b})$ for all $\mathbf{a}, \mathbf{b} \in V$.

(ii) $T(\lambda\mathbf{a}) = \lambda T(\mathbf{a})$ for all $\lambda \in \mathbb{R}$ (or $\mathbb{C}$).

Equivalently, we have $T(\lambda\mathbf{a} + \mu\mathbf{b}) = \lambda T(\mathbf{a}) + \mu T(\mathbf{b})$.

Definition (Image and kernel of map)**.** The *image* of a map $f : U \to V$ is the subset of V $\{f(\mathbf{u}) : \mathbf{u} \in U\}$. The *kernel* is the subset of U $\{\mathbf{u} \in U : f(\mathbf{u}) = \mathbf{0}\}$.

3.3 Rank and nullity

Definition (Rank of linear map)**.** The *rank* of a linear map $f : U \to V$, denoted by $r(f)$, is the dimension of the image of f.

Definition (Nullity of linear map)**.** The *nullity* of f, denoted $n(f)$ is the dimension of the kernel of f.

3.4 Matrices

3.4.1 Examples

3.4.2 Matrix Algebra

Definition (Addition of matrices)**.** Consider two linear maps $\alpha, \beta : \mathbb{R}^n \to \mathbb{R}^m$. The sum of α and β is defined by

$$(\alpha + \beta)(\mathbf{x}) = \alpha(\mathbf{x}) + \beta(\mathbf{x})$$

In terms of the matrix, we have

$$(A + B)_{ij}x_j = A_{ij}x_j + B_{ij}x_j,$$

or

$$(A + B)_{ij} = A_{ij} + B_{ij}.$$

Definition (Scalar multiplication of matrices)**.** Define $(\lambda\alpha)\mathbf{x} = \lambda[\alpha(\mathbf{x})]$. So $(\lambda A)_{ij} = \lambda A_{ij}$.

Definition (Matrix multiplication)**.** Consider maps $\alpha : \mathbb{R}^\ell \to \mathbb{R}^n$ and $\beta : \mathbb{R}^n \to \mathbb{R}^m$. The composition is $\beta\alpha : \mathbb{R}^\ell \to \mathbb{R}^m$. Take $\mathbf{x} \in \mathbb{R}^\ell \mapsto \mathbf{x}'' \in \mathbb{R}^m$. Then $\mathbf{x}'' = (BA)\mathbf{x} = B\mathbf{x}'$, where $\mathbf{x}' = A\mathbf{x}$. Using suffix notation, we have $x''_i = (B\mathbf{x}')_i = b_{ik}x'_k = B_{ik}A_{kj}x_j$. But $x''_i = (BA)_{ij}x_j$. So

$$(BA)_{ij} = B_{ik}A_{kj}.$$

Generally, an $m \times n$ matrix multiplied by an $n \times \ell$ matrix gives an $m \times \ell$ matrix. $(BA)_{ij}$ is given by the ith row of B dotted with the jth column of A.

Definition (Transpose of matrix)**.** If A is an $m \times n$ matrix, the *transpose* A^T is an $n \times m$ matrix defined by $(A^T)_{ij} = A_{ji}$.

Definition (Hermitian conjugate)**.** Define $A^\dagger = (A^T)^*$. Similarly, $(AB)^\dagger = B^\dagger A^\dagger$.

Definition (Symmetric matrix)**.** A matrix is *symmetric* if $A^T = A$.

Definition (Hermitian matrix)**.** A matrix is *Hermitian* if $A^\dagger = A$. (The diagonal of a Hermitian matrix must be real).

Definition (Anti/skew symmetric matrix)**.** A matrix is *anti-symmetric* or *skew symmetric* if $A^T = -A$. The diagonals are all zero.

Definition (Skew-Hermitian matrix)**.** A matrix is *skew-Hermitian* if $A^\dagger = -A$. The diagonals are pure imaginary.

Definition (Trace of matrix)**.** The *trace* of an $n \times n$ matrix A is the sum of the diagonal. $\mathrm{tr}(A) = A_{ii}$.

Definition (Identity matrix)**.** $I = \delta_{ij}$.

3.4.3 Decomposition of an $n \times n$ matrix

3.4.4 Matrix inverse

Definition (Inverse of matrix)**.** Consider an $m \times n$ matrix A and $n \times m$ matrices B and C. If $BA = I$, then we say B is the *left inverse* of A. If $AC = I$, then we say C is the *right inverse* of A. If A is square ($n \times n$), then $B = B(AC) = (BA)C = C$, i.e. the left and right inverses coincide. Both are denoted by A^{-1}, the *inverse* of A. Therefore we have

$$AA^{-1} = A^{-1}A = I.$$

Definition (Invertible matrix)**.** If A has an inverse, then A is *invertible*.

Definition (Orthogonal and unitary matrices)**.** A real $n \times n$ matrix is *orthogonal* if $A^TA = AA^T = I$, i.e. $A^T = A^{-1}$. A complex $n \times n$ matrix is *unitary* if $U^\dagger U = UU^\dagger = I$, i.e. $U^\dagger = U^{-1}$.

3.5 Determinants

3.5.1 Permutations

Definition (Permutation)**.** A *permutation* of a set S is a bijection $\varepsilon : S \to S$.

Notation. Consider the set S_n of all permutations of $1, 2, 3, \cdots, n$. S_n contains $n!$ elements. Consider $\rho \in S_n$ with $i \mapsto \rho(i)$. We write

$$\rho = \begin{pmatrix} 1 & 2 & \cdots & n \\ \rho(1) & \rho(2) & \cdots & \rho(n) \end{pmatrix}.$$

Definition (Fixed point)**.** A *fixed point* of ρ is a k such that $\rho(k) = k$. e.g. in $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 3 & 2 \end{pmatrix}$, 3 is the fixed point. By convention, we can omit the fixed point and write as $\begin{pmatrix} 1 & 2 & 4 \\ 4 & 1 & 2 \end{pmatrix}$.

Definition (Disjoint permutation)**.** Two permutations are *disjoint* if numbers moved by one are fixed by the other, and vice versa. e.g. $\begin{pmatrix} 1 & 2 & 4 & 5 & 6 \\ 5 & 6 & 1 & 4 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 6 \\ 6 & 2 \end{pmatrix} \begin{pmatrix} 1 & 4 & 5 \\ 5 & 1 & 4 \end{pmatrix}$, and the two cycles on the right hand side are disjoint. Disjoint permutations commute, but in general non-disjoint permutations do not.

Definition (Transposition and k-cycle)**.** $\begin{pmatrix} 2 & 6 \\ 6 & 2 \end{pmatrix}$ is a *2-cycle* or a *transposition*, and we can simply write $(2\ 6)$. $\begin{pmatrix} 1 & 4 & 5 \\ 5 & 1 & 4 \end{pmatrix}$ is a 3-cycle, and we can simply write $(1\ 5\ 4)$. (1 is mapped to 5; 5 is mapped to 4; 4 is mapped to 1)

Definition (Sign of permutation)**.** The *sign* of a permutation $\varepsilon(\rho)$ is $(-1)^r$, where r is the number of 2-cycles when ρ is written as a product of 2-cycles. If $\varepsilon(\rho) = +1$, it is an even permutation. Otherwise, it is an odd permutation. Note that $\varepsilon(\rho\sigma) = \varepsilon(\rho)\varepsilon(\sigma)$ and $\varepsilon(\rho^{-1}) = \varepsilon(\rho)$.

Definition (Levi-Civita symbol)**.** The *Levi-Civita* symbol is defined by

$$\varepsilon_{j_1 j_2 \cdots j_n} = \begin{cases} +1 & \text{if } j_1 j_2 j_3 \cdots j_n \text{ is an even permutation of } 1, 2, \cdots n \\ -1 & \text{if it is an odd permutation} \\ 0 & \text{if any 2 of them are equal} \end{cases}$$

Clearly, $\varepsilon_{\rho(1)\rho(2)\cdots\rho(n)} = \varepsilon(\rho)$.

Definition (Determinant)**.** The *determinant* of an $n \times n$ matrix A is defined as:

$$\det(A) = \sum_{\sigma \in S_n} \varepsilon(\sigma) A_{\sigma(1)1} A_{\sigma(2)2} \cdots A_{\sigma(n)n},$$

or equivalently,

$$\det(A) = \varepsilon_{j_1 j_2 \cdots j_n} A_{j_1 1} A_{j_2 2} \cdots A_{j_n n}.$$

3.5.2 Properties of determinants

3.5.3 Minors and Cofactors

Definition (Minor and cofactor)**.** For an $n \times n$ matrix A, define A^{ij} to be the $(n-1) \times (n-1)$ matrix in which row i and column j of A have been removed.

The *minor* of the ijth element of A is $M_{ij} = \det A^{ij}$

The *cofactor* of the ijth element of A is $\Delta_{ij} = (-1)^{i+j} M_{ij}$.

Notation. We use $\bar{\ }$ to denote a symbol which has been missed out of a natural sequence.

4 Matrices and linear equations

4.1 Simple example, 2×2

4.2 Inverse of an $n \times n$ matrix

4.3 Homogeneous and inhomogeneous equations

Definition (Homogeneous equation)**.** If $\mathbf{b} = \mathbf{0}$, then the system is *homogeneous*. Otherwise, it's *inhomogeneous*.

4.3.1 Gaussian elimination

4.4 Matrix rank

Definition (Column and row rank of linear map)**.** The *column rank* of a matrix is the maximum number of linearly independent columns.

The *row rank* of a matrix is the maximum number of linearly independent rows.

4.5 Homogeneous problem $A\mathbf{x} = \mathbf{0}$

4.5.1 Geometrical interpretation

4.5.2 Linear mapping view of $A\mathbf{x} = \mathbf{0}$

4.6 General solution of $A\mathbf{x} = \mathbf{d}$

5 Eigenvalues and eigenvectors

5.1 Preliminaries and definitions

Definition (Multiplicity of root)**.** The root $z = \omega$ has *multiplicity* k if $(z-\omega)^k$ is a factor of $p(z)$ but $(z-\omega)^{k+1}$ is not.

Definition (Eigenvector and eigenvalue)**.** Let $\alpha : \mathbb{C}^n \to \mathbb{C}^n$ be a linear map with associated matrix A. Then $\mathbf{x} \neq \mathbf{0}$ is an *eigenvector* of A if

$$A\mathbf{x} = \lambda\mathbf{x}$$

for some λ. λ is the associated *eigenvalue*. This means that the direction of the eigenvector is preserved by the mapping, but is scaled up by λ.

Definition (Characteristic equation of matrix)**.** The *characteristic equation* of A is

$$\det(A - \lambda I) = 0.$$

Definition (Characteristic polynomial of matrix)**.** The *characteristic polynomial* of A is

$$p_A(\lambda) = \det(A - \lambda I).$$

Definition (Eigenspace)**.** The *eigenspace* denoted by E_λ is the kernel of the matrix $A - \lambda I$, i.e. the set of eigenvectors with eigenvalue λ.

Definition (Algebraic multiplicity of eigenvalue)**.** The *algebraic multiplicity* $M(\lambda)$ or M_λ of an eigenvalue λ is the multiplicity of λ in $p_A(\lambda) = 0$. By the fundamental theorem of algebra,

$$\sum_\lambda M(\lambda) = n.$$

If $M(\lambda) > 1$, then the eigenvalue is *degenerate*.

Definition (Geometric multiplicity of eigenvalue)**.** The *geometric multiplicity* $m(\lambda)$ or m_λ of an eigenvalue λ is the dimension of the eigenspace, i.e. the maximum number of linearly independent eigenvectors with eigenvalue λ.

Definition (Defect of eigenvalue)**.** The *defect* Δ_λ of eigenvalue λ is

$$\Delta_\lambda = M(\lambda) - m(\lambda).$$

It can be proven that $\Delta_\lambda \geq 0$, i.e. the geometric multiplicity is never greater than the algebraic multiplicity.

5.2 Linearly independent eigenvectors

5.3 Transformation matrices

5.3.1 Transformation law for vectors

5.3.2 Transformation law for matrix

5.4 Similar matrices

Definition (Similar matrices)**.** Two $n \times n$ matrices A and B are *similar* if there exists an invertible matrix P such that

$$B = P^{-1}AP,$$

i.e. they represent the same map under different bases. Alternatively, using the language from IA Groups, we say that they are in the same conjugacy class.

5.5 Diagonalizable matrices

Definition (Diagonalizable matrices)**.** An $n \times n$ matrix A is *diagonalizable* if it is similar to a diagonal matrix. We showed above that this is equivalent to saying the eigenvectors form a basis of $\mathbb{C}^n$.

5.6 Canonical (Jordan normal) form

5.7 Cayley-Hamilton Theorem

5.8 Eigenvalues and eigenvectors of a Hermitian matrix

5.8.1 Eigenvalues and eigenvectors

5.8.2 Gram-Schmidt orthogonalization (non-examinable)

5.8.3 Unitary transformation

5.8.4 Diagonalization of $n \times n$ Hermitian matrices

5.8.5 Normal matrices

Definition (Normal matrix)**.** A *normal matrix* as a matrix that commutes with its own Hermitian conjugate, i.e.

$$NN^\dagger = N^\dagger N$$

6 Quadratic forms and conics

Definition (Sesquilinear, Hermitian and quadratic forms)**.** A *sesquilinear form* is a quantity $F = \mathbf{x}^\dagger A\mathbf{x} = x_i^* A_{ij} x_j$. If A is Hermitian, then F is a *Hermitian form.* If A is real symmetric, then F is a *quadratic form.*

6.1 Quadrics and conics

6.1.1 Quadrics

Definition (Quadric)**.** A *quadric* is an n-dimensional surface defined by the zero of a real quadratic polynomial, i.e.

$$\mathbf{x}^T A\mathbf{x} + \mathbf{b}^T\mathbf{x} + c = 0,$$

where A is a real $n \times n$ matrix, $\mathbf{x}, \mathbf{b}$ are n-dimensional column vectors and c is a constant scalar.

6.1.2 Conic sections $(n = 2)$

6.2 Focus-directrix property

Definition (Conic sections)**.** The *eccentricity* and *scale* are properties of a conic section that satisfy the following:

Let the *foci* of a conic section be $(\pm ae, 0)$ and the *directrices* be $x = \pm a/e$.

A *conic section* is the set of points whose distance from focus is $e\times$ distance from directrix which is closer to that of focus (unless $e = 1$, where we take the distance to the other directrix).

7 Transformation groups

7.1 Groups of orthogonal matrices

Definition (Orthogonal group)**.** The *orthogonal group* $O(n)$ is the group of orthogonal matrices.

Definition (Special orthogonal group)**.** The *special orthogonal group* is the subgroup of $O(n)$ that consists of all orthogonal matrices with determinant 1.

7.2 Length preserving matrices

7.3 Lorentz transformations

Definition (Minkowski inner product)**.** The *Minkowski* inner product of 2 vectors $\mathbf{x}$ and $\mathbf{y}$ is

$$\langle \mathbf{x} \mid \mathbf{y} \rangle = \mathbf{x}^T J \mathbf{y},$$

where

$$J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Then $\langle \mathbf{x} \mid \mathbf{y} \rangle = x_1 y_1 - x_2 y_2$.

Definition (Preservation of inner product)**.** A transformation matrix M preserves the Minkowski inner product if

$$\langle \mathbf{x} | \mathbf{y} \rangle = \langle M\mathbf{x} | M\mathbf{y} \rangle$$

for all $\mathbf{x}, \mathbf{y}$.

Definition (Lorentz matrix)**.** A *Lorentz matrix* or a *Lorentz boost* is a matrix in the form

$$B_v = \frac{1}{\sqrt{1 - v^2}} \begin{pmatrix} 1 & v \\ v & 1 \end{pmatrix}.$$

Here $|v| < 1$, where we have chosen units in which the speed of light is equal to 1. We have $B_v = H_{\tanh^{-1} v}$

Definition (Lorentz group)**.** The *Lorentz group* is a group of all Lorentz matrices under matrix multiplication.