Numbers and Sets (2014–15) Example Sheet 1 of 4

1. For which $n \in \mathbb{N}$, if any, are three numbers of the form $n, n+2, n+4$ all prime?

2. Between 0 and 10 there are four primes. Another example of two consecutive multiples of ten, between which there are four primes, is 10 and 20. Are there further examples?

3. If n^2 is a multiple of 3, must n be a multiple of 3?

4. Write down the negations of the following assertions (where $m, n, a, b \in \mathbb{N}$):

(i) if Coke is not worse than Pepsi then Osborne hasn't a clue what he's about.

(ii) $\forall m\ \exists n\ \forall a\ \forall b\ \ (n \geq m)\ \wedge\ [(a = 1) \vee (b = 1) \vee (ab \neq n)]$,

5. The sum of some (not necessarily distinct) natural numbers is 100. How large can their product be?

6. Prove that $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$.

7. The *symmetric difference* $A \triangle B$ of two sets A and B is the set of elements that belong to exactly one of A and B. Express this in terms of $\cup$, $\cap$ and $\setminus$. Prove that $\triangle$ is associative.

8. Let $A_1, A_2, A_3, \ldots$ be sets such that $A_1 \cap A_2 \cap \ldots \cap A_n \neq \emptyset$ holds for all n. Must it be that $\bigcap_{n=1}^{\infty} A_n \neq \emptyset$?

9. Prove that $f \circ g$ is injective if f and g are injective. Does $f \circ g$ injective imply f injective? Does it imply g injective? What if we replace 'injective' by 'surjective' *passim*?

10. Let $A = \{1, 2, 3\}$ and $B = \{1, 2, 3, 4, 5\}$? How many functions $A \to B$ are there? How many are injections? Count the number of surjections $B \to A$.

11. Let $f : X \to Y$ and let $C, D \subset Y$. Prove that $f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D)$. Let $A, B \subset X$. Must it be true that $f(A \cap B) = f(A) \cap f(B)$?

12. Define a relation R on $\mathbb{N}$ by setting aRb if $a \mid b$ or $b \mid a$. Is R an equivalence relation?

13. The relation S *contains* the relation R if aSb whenever aRb. Let R be the relation on $\mathbb{Z}$ 'aRb if $b = a + 3$'. How many equivalence relations on $\mathbb{Z}$ contain R?

14. Construct a function $f : \mathbb{R} \to \mathbb{R}$ that takes every value on every interval — in other words, for every $a < b$ and every c there is an x with $a < x < b$ such that $f(x) = c$.

15. Find a bijection $f : \mathbb{Q} \to \mathbb{Q} \setminus \{0\}$. Can f be strictly increasing (that is, $f(x) < f(y)$ whenever $x < y$)?

1. Find the highest common factor of 12345 and 54321. Find $u, v \in \mathbb{Z}$ with $76u + 45v = 1$. Does $3381x + 2646y = 21$ have an integer solution?

2. Find the convergents to the fraction $\frac{57}{44}$. Prove that if x and y are integers such that $57x + 44y = 1$, then $x = 17 - 44k$ and $y = 57k - 22$ for some $k \in \mathbb{Z}$.

3. Let $a, b, c \in \mathbb{N}$. Must the numbers $(a, b)(c, d)$ and (ac, bd) be equal? If not, must one be a factor of the other? If $(a, b) = (a, c) = 1$, must we have $(a, bc) = 1$?

4. Show that, for any $a,\ b \in \mathbb{N}$, the number $\ell = ab/(a, b)$ is an integer (called the *least common multiple* of a and b). Show also that ℓ is divisible by both a and b, and that if $n \in \mathbb{N}$ is divisible by both a and b then $\ell \mid n$.

5. Do there exist 100 consecutive natural numbers none of which is prime?

6. In the sequence 41, 43, 47, 53, 61, …, each difference is two more than the previous one. Are all the numbers in the sequence prime?

7. Let A be a set of n positive integers. Show that every sequence of 2^n numbers taken from A contains a consecutive block of numbers whose product is a square. (For instance, 2,5,3,2,5,2,3,5 contains the block 5,3,2,5,2,3.)

8. Use the inclusion-exclusion principle to count the number of primes less than 121.

9. How many subsets of $\{1, 2, \ldots, n\}$ are there of even size?

10. By suitably interpreting each side, or otherwise, establish the identities

$$\binom{k}{k} + \binom{k+1}{k} + \binom{k+2}{k} + \cdots + \binom{n}{k} = \binom{n+1}{k+1}$$

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n}$$

11. A *triomino* is an L-shaped pattern made from three square tiles. A $2^k \times 2^k$ chessboard, whose squares are the same size as the tiles, has one of its squares painted puce. Show that the chessboard can be covered with triominoes so that only the puce square is exposed.

12. By considering the number of ways to partition a set of order $2n$ into n parts of order 2, show that $(n+1)(n+2)\ldots(2n)$ is divisible by 2^n but not by 2^{n+1}.

13. The *repeat* of a natural number is obtained by writing it twice in a row (for example, the repeat of 356 is 356356). Is there a number whose repeat is a perfect square?

14. Let $a < b$ be distinct natural numbers. Prove that every block of b consecutive natural numbers contains two distinct numbers whose product is divisible by ab. Suppose now $a < b < c$. Must every block of c consecutive numbers contain three distinct numbers whose product is divisible by abc?

1. Show that a number is divisible by 9 if, and only if, the sum of its digits is divisible by 9.

2. The *Fibonacci numbers* $F_0, F_1, F_2 \ldots$ are defined by $F_0 = 0$, $F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for all $n \geq 2$. Is F_{2010} even or odd? Is it a multiple of 3?

Show (by induction on k or otherwise) that $F_{n+k} = F_k F_{n+1} + F_{k-1} F_n$ for $k \geq 1$. Deduce that $(F_m, F_n) = (F_{m-n}, F_n)$, and thence that $(F_m, F_n) = F_{(m,n)}$.

3. Let p be prime. Prove that if $0 < k < p$ then $\binom{p}{k} \equiv 0 \pmod{p}$. If you do this by using a formula for $\binom{p}{k}$ then argue correctly. Can you give a proof directly from the definition?

4. Solve these congruences:-

(i) $77x \equiv 11 \pmod{40}$, (ii) $12y \equiv 30 \pmod{54}$,

(iii) $z \equiv 13 \pmod{21}$ and $3z \equiv 2 \pmod{17}$ simultaneously.

5. Do there exist 100 consecutive natural numbers, each of which has a proper square factor?

6. Show that the exponent of the prime p in the prime factorisation of $n!$ is $\sum_{i \geq 1} \lfloor n/p^i \rfloor$, where $\lfloor x \rfloor$ denotes the integer part of x. Prove that this equals $(n - S_n)/(p-1)$, where S_n is the sum of the digits in the base p representation of n. Evaluate $1000! \pmod{10^{249}}$.

7. Without using a calculator, evaluate $20! 21^{20} \pmod{23}$ and $17^{10000} \pmod{30}$.

8. By considering the n fractions $\frac{1}{n}, \frac{2}{n}, \ldots, \frac{n}{n}$, or otherwise, prove that $n = \sum_{d|n} \varphi(d)$.

9. An RSA encryption scheme (n, e) has modulus $n = 187$ and encoding exponent $e = 7$. Find a suitable decoding exponent d. Check your answer by encoding the number 35 and then decoding the result. *(Remember, no calculators!)*

10. Let p be a prime of the form $3k+2$. Show that if $x^3 \equiv 1 \pmod{p}$ then $x \equiv 1 \pmod{p}$. Deduce that every number is a cube (mod p): i.e., $y^3 \equiv a \pmod{p}$ is soluble for all $a \in \mathbb{Z}$. Is the same ever true if p is of the form $3k+1$?

11. Using the least upper bound axiom, prove that there is a real number x satisfying $x^3 = 2$.

12. Prove that $\sqrt{2} + \sqrt{3}$ is irrational and algebraic. Do the same for $2^{1/3} + 2^{2/3}$.

13. Suppose that $x \in \mathbb{R}$ and $x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_0 = 0$, where $a_{n-1}, \ldots, a_0 \in \mathbb{Z}$. Prove that either x is an integer or it is irrational.

14. Show that $\sqrt[100]{\sqrt{3}+\sqrt{2}} + \sqrt[100]{\sqrt{3}-\sqrt{2}}$ is irrational.

15. Show that $a^4 + b^7 = 11^{11}$ has no solution with $a, b \in \mathbb{Z}$.

Numbers and Sets (2014–15) Example Sheet 4 of 4

1. Define a sequence $(x_n)_{n=1}^{\infty}$ by setting $x_1 = 1$ and $x_{n+1} = \frac{x_n}{1+\sqrt{x_n}}$ for all $n \geq 1$. Show that $(x_n)_{n=1}^{\infty}$ converges, and determine its limit.

2. Let $(a_n)_{n=1}^{\infty}$ be a sequence of reals. Show that if $(a_n)_{n=1}^{\infty}$ is convergent then we must have $a_n - a_{n-1} \to 0$. If $a_n - a_{n-1} \to 0$, must $(a_n)_{n=1}^{\infty}$ be convergent?

3. Let $[a_n, b_n]$, $n = 1, 2, \ldots$, be closed intervals with $[a_n, b_n] \cap [a_m, b_m] \neq \emptyset$ for all n, m. Prove that $\bigcap_{n=1}^{\infty}[a_n, b_n] \neq \emptyset$.

4. Which of the following sequences $(x_n)_{n=1}^{\infty}$ converge?

$$x_n = \frac{3n}{n+3} \qquad x_n = \frac{n^{100}}{2^n} \qquad x_n = \sqrt{n+1} - \sqrt{n} \qquad x_n = (n!)^{1/n}$$

5. Which of the following series converge?

$$\sum_{n=1}^{\infty} \frac{1}{1+n^2} \qquad \sum_{n=1}^{\infty} \frac{n!}{n^n} \qquad \sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+n}} \qquad \sum_{n=1}^{\infty}{}^{*} \frac{1}{n}$$

In the last case, the $*$ means omit all values of n which, when written in base 10, have some digit equal to 7.

6. Let $a_n \in \mathbb{R}$ and let $b_n = \frac{1}{n}\sum_{i=1}^{n} a_i$. Show that, if $a_n \to a$ as $n \to \infty$, then $b_n \to a$ also.

7. Let $\sum_{n=1}^{\infty} x_n$ be a divergent series, where $x_n > 0$ for all n. Show that there is a divergent series $\sum_{n=1}^{\infty} y_n$ with $y_n > 0$ for all n, such that $y_n/x_n \to 0$.

8. A real number $r = 0 \cdot d_1 d_2 d_3 \ldots$ is called *repetitive* if its decimal expansion contains arbitrarily long blocks that are the same; that is, for every k there exist distinct m and n such that $d_m = d_n,\ d_{m+1} = d_{n+1},\ \ldots,\ d_{m+k} = d_{n+k}$. Prove that the square of a repetitive number is repetitive.

9. Show that any collection of pairwise disjoint discs in the plane is countable. What happens if we replace 'discs' by 'circles'?

10. Show that the collection of all finite subsets of $\mathbb{N}$ is countable. What goes wrong if we try to use the diagonal argument to show that it is uncountable?

11. A function $f : \mathbb{N} \to \mathbb{N}$ is *increasing* if $f(n+1) \geq f(n)$ for all n and *decreasing* if $f(n+1) \leq f(n)$ for all n. Is the set of increasing functions countable or uncountable? What about the set of decreasing functions?

12. Find an injection $\mathbb{R}^2 \to \mathbb{R}$. Is there an injection from the set of all real sequences to $\mathbb{R}$?

13. Let $\sum_{n=1}^{\infty} x_n$ be convergent. If $x_n > 0$ for all n, show that $\sum_{n=1}^{\infty} x_n^2$ also converges. What if sometimes $x_n < 0$? What are the corresponding answers for $\sum_{n=1}^{\infty} x_n^3$?

14. Let $S \subset \mathcal{P}\mathbb{N}$ be such that if $A, B \in S$ then $A \subset B$ or $B \subset A$. Can S be uncountable? Is there an uncountable family $T \subset \mathcal{P}\mathbb{N}$ such that $A \cap B$ is finite for all distinct $A, B \in T$?

15. Is there an enumeration of $\mathbb{Q}$ as $q_1, q_2, q_3, \ldots$ such that $\sum (q_n - q_{n+1})^2$ converges?