

Part IA — Differential Equations

Theorems with proof

Based on lectures by M. G. Worster

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These notes are not endorsed by the lecturers, and I have modified them (often significantly) after lectures. They are nowhere near accurate representations of what was actually lectured, and in particular, all errors are almost surely mine.

Basic calculus

Informal treatment of differentiation as a limit, the chain rule, Leibnitz's rule, Taylor series, informal treatment of O and o notation and l'Hôpital's rule; integration as an area, fundamental theorem of calculus, integration by substitution and parts. [3]

Informal treatment of partial derivatives, geometrical interpretation, statement (only) of symmetry of mixed partial derivatives, chain rule, implicit differentiation. Informal treatment of differentials, including exact differentials. Differentiation of an integral with respect to a parameter. [2]

First-order linear differential equations

Equations with constant coefficients: exponential growth, comparison with discrete equations, series solution; modelling examples including radioactive decay.

Equations with non-constant coefficients: solution by integrating factor. [2]

Nonlinear first-order equations

Separable equations. Exact equations. Sketching solution trajectories. Equilibrium solutions, stability by perturbation; examples, including logistic equation and chemical kinetics. Discrete equations: equilibrium solutions, stability; examples including the logistic map. [4]

Higher-order linear differential equations

Complementary function and particular integral, linear independence, Wronskian (for second-order equations), Abel's theorem. Equations with constant coefficients and examples including radioactive sequences, comparison in simple cases with difference equations, reduction of order, resonance, transients, damping. Homogeneous equations. Response to step and impulse function inputs; introduction to the notions of the Heaviside step-function and the Dirac delta-function. Series solutions including statement only of the need for the logarithmic solution. [8]

Multivariate functions: applications

Directional derivatives and the gradient vector. Statement of Taylor series for functions on \mathbb{R}^n . Local extrema of real functions, classification using the Hessian matrix. Coupled first order systems: equivalence to single higher order equations; solution by matrix methods. Non-degenerate phase portraits local to equilibrium points; stability.

Simple examples of first- and second-order partial differential equations, solution of the wave equation in the form $f(x + ct) + g(x - ct)$. [5]

Contents

0	Introduction	5
1	Differentiation	6
1.1	Differentiation	6
1.2	Small o and big O notations	6
1.3	Methods of differentiation	6
1.4	Taylor's theorem	6
1.5	L'Hopital's rule	7
2	Integration	8
2.1	Integration	8
2.2	Methods of integration	8
3	Partial differentiation	9
3.1	Partial differentiation	9
3.2	Chain rule	9
3.3	Implicit differentiation	9
3.4	Differentiation of an integral wrt parameter in the integrand	9
4	First-order differential equations	10
4.1	The exponential function	10
4.2	Homogeneous linear ordinary differential equations	10
4.3	Forced (inhomogeneous) equations	10
4.3.1	Constant forcing	10
4.3.2	Eigenfunction forcing	10
4.4	Non-constant coefficients	10
4.5	Non-linear equations	10
4.5.1	Separable equations	10
4.5.2	Exact equations	10
4.6	Solution curves (trajectories)	10
4.7	Fixed (equilibrium) points and stability	10
4.7.1	Perturbation analysis	10
4.7.2	Autonomous systems	10
4.7.3	Logistic Equation	10
4.8	Discrete equations (Difference equations)	10
5	Second-order differential equations	11
5.1	Constant coefficients	11
5.1.1	Complementary functions	11
5.1.2	Second complementary function	11
5.1.3	Phase space	11
5.2	Particular integrals	11
5.2.1	Guessing	11
5.2.2	Resonance	11
5.2.3	Variation of parameters	11
5.3	Linear equidimensional equations	11
5.4	Difference equations	11
5.5	Transients and damping	11

5.6	Impulses and point forces	11
5.6.1	Dirac delta function	11
5.7	Heaviside step function	11
6	Series solutions	12
7	Directional derivative	13
7.1	Gradient vector	13
7.2	Stationary points	13
7.3	Taylor series for multi-variable functions	13
7.4	Classification of stationary points	13
7.5	Contours of $f(x, y)$	13
8	Systems of differential equations	14
8.1	Linear equations	14
8.2	Nonlinear dynamical systems	14
9	Partial differential equations (PDEs)	15
9.1	First-order wave equation	15
9.2	Second-order wave equation	15
9.3	The diffusion equation	15

0 Introduction

1 Differentiation

1.1 Differentiation

1.2 Small o and big O notations

Proposition.

$$f(x_0 + h) = f(x_0) + f'(x_0)h + o(h)$$

Proof. We have

$$f'(x_0) = \frac{f(x_0 + h) - f(x_0)}{h} + \frac{o(h)}{h}$$

by the definition of the derivative and the small o notation. The result follows. \square

1.3 Methods of differentiation

Theorem (Chain rule). Given $f(x) = F(g(x))$, then

$$\frac{df}{dx} = \frac{dF}{dg} \frac{dg}{dx}.$$

Proof. Assuming that $\frac{dg}{dx}$ exists and is therefore finite, we have

$$\begin{aligned} \frac{df}{dx} &= \lim_{h \rightarrow 0} \frac{F(g(x+h)) - F(g(x))}{h} \\ &= \lim_{h \rightarrow 0} \frac{F[g(x) + hg'(x) + o(h)] - F(g(x))}{h} \\ &= \lim_{h \rightarrow 0} \frac{F(g(x)) + (hg'(x) + o(h))F'(g(x)) + o(hg'(x) + o(h)) - F(g(x))}{h} \\ &= \lim_{h \rightarrow 0} g'(x)F'(g(x)) + \frac{o(h)}{h} \\ &= g'(x)F'(g(x)) \\ &= \frac{dF}{dg} \frac{dg}{dx} \end{aligned} \quad \square$$

Theorem (Product Rule). Give $f(x) = u(x)v(x)$. Then

$$f'(x) = u'(x)v(x) + u(x)v'(x).$$

Theorem (Leibniz's Rule). Given $f = uv$, then

$$f^{(n)}(x) = \sum_{r=0}^n \binom{n}{r} u^{(r)}v^{(n-r)},$$

where $f^{(n)}$ is the n -th derivative of f .

1.4 Taylor's theorem

Theorem (Taylor's Theorem). For n -times differentiable f , we have

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \cdots + \frac{h^n}{n!}f^{(n)}(x) + E_n,$$

where $E_n = o(h^n)$ as $h \rightarrow 0$. If $f^{(n+1)}$ exists, then $E_n = O(h^{n+1})$.

1.5 L'Hopital's rule

Theorem (L'Hopital's Rule). Let $f(x)$ and $g(x)$ be differentiable at x_0 , and $\lim_{x \rightarrow x_0} f(x) = \lim_{x \rightarrow x_0} g(x) = 0$. Then

$$\lim_{x \rightarrow x_0} \frac{f(x)}{g(x)} = \lim_{x \rightarrow x_0} \frac{f'(x)}{g'(x)}.$$

Proof. From the Taylor's Theorem, we have $f(x) = f(x_0) + (x - x_0)f'(x_0) + o(x - x_0)$, and similarly for $g(x)$. Thus

$$\begin{aligned} \lim_{x \rightarrow x_0} \frac{f(x)}{g(x)} &= \lim_{x \rightarrow x_0} \frac{f(x_0) + (x - x_0)f'(x_0) + o(x - x_0)}{g(x_0) + (x - x_0)g'(x_0) + o(x - x_0)} \\ &= \lim_{x \rightarrow x_0} \frac{f'(x_0) + \frac{o(x - x_0)}{x - x_0}}{g'(x_0) + \frac{o(x - x_0)}{x - x_0}} \\ &= \lim_{x \rightarrow x_0} \frac{f'(x)}{g'(x)} \quad \square \end{aligned}$$

2 Integration

2.1 Integration

Theorem (Fundamental Theorem of Calculus). Let $F(x) = \int_a^x f(t) dt$. Then $F'(x) = f(x)$.

Proof.

$$\begin{aligned}\frac{d}{dx}F(x) &= \lim_{h \rightarrow 0} \frac{1}{h} \left[\int_a^{x+h} f(t) dt - \int_a^x f(t) dt \right] \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \int_x^{x+h} f(t) dt \\ &= \lim_{h \rightarrow 0} \frac{1}{h} [f(x)h + O(h^2)] \\ &= f(x)\end{aligned}$$

□

2.2 Methods of integration

Theorem (Integration by parts).

$$\int uv' dx = uv - \int vu' dx.$$

Proof. From the product rule, we have $(uv)' = uv' + u'v$. Integrating the whole expression and rearranging gives the formula above. □

3 Partial differentiation

3.1 Partial differentiation

Theorem. If f has continuous second partial derivatives, then $f_{xy} = f_{yx}$.

3.2 Chain rule

Theorem (Chain rule for partial derivatives).

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy.$$

Given this form, we can integrate the differentials to obtain the integral form:

$$\int df = \int \frac{\partial f}{\partial x} dx + \int \frac{\partial f}{\partial y} dy,$$

or divide by another small quantity. e.g. to find the slope along the path $(x(t), y(t))$, we can divide by dt to obtain

$$\frac{df}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}.$$

3.3 Implicit differentiation

Theorem (Multi-variable implicit differentiation). Given an equation

$$F(x, y, z) = c$$

for some constant c , we have

$$\left. \frac{\partial z}{\partial x} \right|_y = - \frac{(\partial F)/(\partial x)}{(\partial F)/(\partial z)}$$

Proof.

$$\begin{aligned} dF &= \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy + \frac{\partial F}{\partial z} dz \\ \left. \frac{\partial F}{\partial x} \right|_y &= \frac{\partial F}{\partial x} \left. \frac{\partial x}{\partial x} \right|_y + \frac{\partial F}{\partial y} \left. \frac{\partial y}{\partial x} \right|_y + \frac{\partial F}{\partial z} \left. \frac{\partial z}{\partial x} \right|_y = 0 \\ \frac{\partial F}{\partial x} + \frac{\partial F}{\partial z} \left. \frac{\partial z}{\partial x} \right|_y &= 0 \\ \left. \frac{\partial z}{\partial x} \right|_y &= - \frac{(\partial F)/(\partial x)}{(\partial F)/(\partial z)} \quad \square \end{aligned}$$

3.4 Differentiation of an integral wrt parameter in the integrand

Theorem (Differentiation under the integral sign).

$$\frac{d}{dx} \int_0^{b(x)} f(y, c(x)) dy = f(b, c)b'(x) + c'(x) \int_0^b \frac{\partial f}{\partial c} dy$$

4 First-order differential equations

4.1 The exponential function

4.2 Homogeneous linear ordinary differential equations

Theorem. Any linear, homogeneous, ordinary differential equation with constant coefficients has solutions of the form e^{mx} .

4.3 Forced (inhomogeneous) equations

4.3.1 Constant forcing

4.3.2 Eigenfunction forcing

4.4 Non-constant coefficients

4.5 Non-linear equations

4.5.1 Separable equations

4.5.2 Exact equations

Theorem. If $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ through a simply-connected domain \mathcal{D} , then $P dx + Q dy$ is an exact differential of a single-valued function in \mathcal{D} .

4.6 Solution curves (trajectories)

4.7 Fixed (equilibrium) points and stability

4.7.1 Perturbation analysis

4.7.2 Autonomous systems

4.7.3 Logistic Equation

4.8 Discrete equations (Difference equations)

5 Second-order differential equations

5.1 Constant coefficients

5.1.1 Complementary functions

5.1.2 Second complementary function

5.1.3 Phase space

Theorem (Abel's Theorem). Given an equation $y'' + p(x)y' + q(x)y = 0$, either $W = 0$ for all x , or $W \neq 0$ for all x . i.e. iff two solutions are independent for some particular x , then they are independent for all x .

Proof. If y_1 and y_2 are both solutions, then

$$\begin{aligned}y_2(y_1'' + py_1' + qy_1) &= 0 \\y_1(y_2'' + py_2' + qy_2) &= 0\end{aligned}$$

Subtracting the two equations, we have

$$y_1y_2'' - y_2y_1'' + p(y_1y_2' - y_2y_1') = 0$$

Note that $W = y_1y_2' - y_2y_1'$ and $W' = y_1y_2'' + y_1'y_2' - (y_2'y_1' + y_2y_1'') = y_1y_2'' - y_2y_1''$

$$\begin{aligned}W' + P(x)W &= 0 \\W(x) &= W_0e^{-\int P dx},\end{aligned}$$

Where $W_0 = \text{const.}$ Since the exponential function is never zero, either $W_0 = 0$, in which case $W = 0$, or $W_0 \neq 0$ and $W \neq 0$ for any value of x . \square

5.2 Particular integrals

5.2.1 Guessing

5.2.2 Resonance

5.2.3 Variation of parameters

5.3 Linear equidimensional equations

5.4 Difference equations

5.5 Transients and damping

5.6 Impulses and point forces

5.6.1 Dirac delta function

5.7 Heaviside step function

6 Series solutions

7 Directional derivative

7.1 Gradient vector

7.2 Stationary points

7.3 Taylor series for multi-variable functions

7.4 Classification of stationary points

Proposition. H is positive definite if and only if the signature is $+, +, \dots, +$.
 H is negative definite if and only if the signature is $-, +, \dots, (-1)^n$. Otherwise,
 H is indefinite.

7.5 Contours of $f(x, y)$

8 Systems of differential equations

8.1 Linear equations

8.2 Nonlinear dynamical systems

9 Partial differential equations (PDEs)

9.1 First-order wave equation

9.2 Second-order wave equation

9.3 The diffusion equation