Mathematical Tripos Part IA

M. G. Worster

Differential Equations A3

Michaelmas 2014

DIFFERENTIAL EQUATIONS

Examples Sheet 1

The starred questions are intended as extras: do them if you have time, but not at the expense of unstarred questions on later sheets.

1. Show, from first principles, that, for non-negative integer n $\dfrac{d}{dx}x^n = nx^{n-1}$.

2. Let $f(x) = u(x)v(x)$. Use the definition of the derivative of a function to show that
$$\frac{df}{dx} = u\frac{dv}{dx} + \frac{du}{dx}v.$$

3. Calculate
   (i) $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(e^{-x^2}\sin 2x\right)$,
   (ii) $\dfrac{\mathrm{d}^{12}}{\mathrm{d}x^{12}}(x\cos x)$ using (a) the Leibniz rule and (b) repeated application of the product rule,
   (iii) $\dfrac{\mathrm{d}^5}{\mathrm{d}x^5}\left([\ln(x)]^2\right)$.

4. (i) Write down or determine the Taylor series for $f(x) = e^{ax}$ about $x = 1$.
   (ii) Write down or determine the Taylor series for $\ln(1+x)$ about $x = 0$. Then show that
$$\lim_{k\to\infty} k\ln(1 + x/k) = x$$
   and deduce that
$$\lim_{k\to\infty}(1 + x/k)^k = e^x.$$
   What property of the exponential function did you need?

5. Determine by any method the first three non-zero terms of the Taylor expansions about $x = 0$ of
   (i) $(x^2 + a)^{-3/2}$,
   (ii) $\ln(\cos x)$,
   iii) $\exp\left\{-\frac{1}{(x-a)^2}\right\}$,
   where a is a constant.

6. By considering the area under the curves $y = \ln x$ and $y = \ln(x-1)$, show that
$$N \ln N - N < \ln(N!) < (N+1)\ln(N+1) - N.$$
Hence show that
$$|\ln N! - N \ln N + N| < \ln\left(1 + \frac{1}{N}\right)^N + \ln(1+N).$$

7. Show that $y(x) = \int_x^\infty e^{-t^2} dt$ satisfies the differential equation $y'' + 2xy' = 0$.

*8. Let J_n be the indefinite integral
$$J_n = \int \frac{x^{-n}\,\mathrm{d}x}{(ax^2 + 2bx + c)^{\frac{1}{2}}} \;.$$
By integrating $\int x^{-n-1}(ax^2 + 2bx + c)^{\frac{1}{2}}\,\mathrm{d}x$ by parts, show that for $n \neq 0$,
$$ncJ_{n+1} + (2n-1)bJ_n + (n-1)aJ_{n-1} = -x^{-n}(ax^2 + 2bx + c)^{\frac{1}{2}}.$$
Hence evaluate
$$\int_1^2 \frac{\mathrm{d}x}{x^{5/2}(x+2)^{\frac{1}{2}}}.$$

*9. In a large population, the proportion with income between x and $x + \mathrm{d}x$ is $f(x)\mathrm{d}x$. Express the mean (average) income μ as an integral, assuming that any positive income is possible.

Let $p = F(x)$ be the proportion of the population with income less than x, and $G(x)$ be the mean (average) income earned by people with income less than x. Further, let $\theta(p)$ be the proportion of the total income which is earned by people with income less than x as a function of the proportion p of the population which has income less than x. Express $F(x)$ and $G(x)$ as integrals and thence derive an expression for $\theta(p)$, showing that

$$\theta(0) = 0, \quad \theta(1) = 1$$

and

$$\theta'(p) = \frac{F^{-1}(p)}{\mu}, \quad \theta''(p) = \frac{1}{\mu f(F^{-1}(p))} > 0.$$

Sketch the graph of a function $\theta(p)$ with these properties and deduce that unless there is complete equality of income distribution, the bottom (in terms of income) $100p\%$ of the population receive less than $100p\%$ of the total income, for all positive values of p.

10. For $f(x, y) = \exp(-xy)$, find $(\partial f/\partial x)_y$ and $(\partial f/\partial y)_x$. Check that $\dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{\partial^2 f}{\partial y \partial x}$. Find $(\partial f/\partial r)_\theta$ and $(\partial f/\partial \theta)_r$,

(i) using the chain rule,

(ii) by first expressing f in terms of the polar coordinates r, θ,

and check that the two methods give the same results.

[Recall: $x = r\cos\theta, \quad y = r\sin\theta$.]

11. If $xyz + x^3 + y^4 + z^5 = 0$ (an implicit equation for any of the variables x, y, z in terms of the other two), find

$$\left(\frac{\partial x}{\partial y}\right)_z, \quad \left(\frac{\partial y}{\partial z}\right)_x, \quad \left(\frac{\partial z}{\partial x}\right)_y$$

and show that their product is -1.

Does this result hold for an arbitrary relation $f(x, y, z) = 0$?

What about $f(x_1, x_2, \cdots, x_n) = 0$?

12. In thermodynamics, the pressure of a system, p, can be considered as a function of the variables V (volume) and T (temperature) or as a function of the variables V and S (entropy).

(i) By expressing $p(V,S)$ in the form $p(V,S(V,T))$ evaluate

$$\left(\frac{\partial p}{\partial V}\right)_T - \left(\frac{\partial p}{\partial V}\right)_S \text{ in terms of } \left(\frac{\partial S}{\partial V}\right)_T \text{ and } \left(\frac{\partial S}{\partial p}\right)_V .$$

(ii) Hence, using $TdS = dU + pdV$ (conservation of energy with U the internal energy), show that

$$\left(\frac{\partial \mathrm{ln} p}{\partial \mathrm{ln} V}\right)_T - \left(\frac{\partial \mathrm{ln} p}{\partial \mathrm{ln} V}\right)_S = \left(\frac{\partial (pV)}{\partial T}\right)_V \left[\frac{p^{-1}(\partial U/\partial V)_T + 1}{(\partial U/\partial T)_V}\right] .$$

$\left[\text{Hint: } \left(\frac{\partial \ln p}{\partial \ln V}\right)_T = \frac{V}{p}\left(\frac{\partial p}{\partial V}\right)_T\right]$

13. By differentiating I with respect to λ, show that

$$I(\lambda,\alpha) = \int_0^\infty \frac{\sin \lambda x}{x} e^{-\alpha x} dx = \tan^{-1}\frac{\lambda}{\alpha} + c(\alpha).$$

Show that $c(\alpha)$ is constant (independent of α) and hence, by considering the limits $\alpha \to \infty$ and $\alpha \to 0$, show that, if $\lambda > 0$,

$$\int_0^\infty \frac{\sin \lambda x}{x} dx = \frac{\pi}{2}.$$

14. Let $f(x) = \left[\int_0^x e^{-t^2}\mathrm{d}t\right]^2$ and let $g(x) = \int_0^1 [e^{-x^2(t^2+1)}/(1+t^2)]\mathrm{d}t$.

Show that

$$f'(x) + g'(x) = 0.$$

Deduce that

$$f(x) + g(x) = \pi/4,$$

and hence that

$$\int_0^\infty e^{-t^2}\mathrm{d}t = \frac{\sqrt{\pi}}{2}.$$

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Examples Sheet 2

The starred questions are intended as extras: do them if you have time, but not at the expense of unstarred questions on earlier sheets

1. According to Newton's law of cooling, the rate of change of the temperature of an object is proportional to the difference in temperature between the object and its surroundings. A forensic scientist enters a crime scene at 5:00 pm and discovers a cup of tea at temperature 40°C. At 5:30 pm its temperature is only 30°C. Giving all details of the mathematical methodology employed and assumptions made, estimate the time at which the tea was made.

2. Determine the half-life of Thorium-234 if a sample of 5 grams is reduced to 4 grams in one week. What amount of Thorium is left after three months?

3. Find the solutions of the initial value problems

   (i) $y' + 2y = e^{-x}$, $y(0) = 1$;

   (ii) $y' - y = 2xe^{2x}$, $y(0) = 1$.

4. Show that the general solution of

$$y' - y = e^{ux} \ , \ u \neq 1 \ , \qquad (*)$$

   can be written (by means of a suitable choice of A) in the form

$$y(x) = Ae^x + \frac{e^{ux} - e^x}{u - 1} \ .$$

   By taking the limit as $u \to 1$ and using l'Hôpital's rule, find the general solution of (*) when $u = 1$.

5. Solve

   (i) $y'x \sin x + (\sin x + x \cos x)y = xe^x$;

   (ii) $y' \tan x + y = 1$;

   (iii) $y' = (e^y - x)^{-1}$.

6. Find the general solutions of

   (i) $y' = x^2(1 + y^2)$,

   (ii) $y' = \cos^2 x \cos^2 2y$,

   (iii) $y' = (x - y)^2$,

   (iv) $(\mathrm{e}^y + x)y' + (\mathrm{e}^x + y) = 0$.

7. Find all solutions of the equation

$$y\frac{\mathrm{d}y}{\mathrm{d}x} - x = 0 ,$$

   and give a sketch showing the solutions. By means of the substitution $y = \log u - x$, deduce the general solution of

$$(\log u - x)\frac{\mathrm{d}u}{\mathrm{d}x} - u \log u = 0 .$$

   Sketch the solutions, starting from your previous sketch and drawing first the lines to which $y = \pm x$ are mapped.

8. In each of the following sketch a few solution curves. It might help you to consider values of y' on the axes, or contours of constant y', or the asymptotic behaviour when y is large.

   (i) $y' + xy = 1$,

   (ii) $y' = x^2 + y^2$,

   (iii) $y' = (1 - y)(2 - y)$.

9. (i) Sketch the solution curves for the equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = xy .$$

   Find the family of solutions determined by this equation and reassure yourself that your sketches were appropriate.

   (ii) Sketch the solution curves for the equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x - y}{x + y} .$$

   By rewriting the equation in the form

$$\left(x\frac{\mathrm{d}y}{\mathrm{d}x} + y\right) + y\frac{\mathrm{d}y}{\mathrm{d}x} = x ,$$

   find and sketch the family of solutions.

   *Does the substitution $y = ux$ lead to an easier method of solving this equation?

10. Measurements on a yeast culture have shown that the rate of increase of the amount, or 'biomass', of yeast is related to the biomass itself by the equation

$$\frac{\mathrm{d}N}{\mathrm{d}t} = aN - bN^2 \ ,$$

where $N(t)$ is a measure of the biomass at time t, and a and b are positive constants. Without solving the equation, find in terms of a and b:

(i) the value of N at which $\mathrm{d}N/\mathrm{d}t$ is a maximum;

(ii) the values of N at which $\mathrm{d}N/\mathrm{d}t$ is zero, and the corresponding values of $\mathrm{d}^2N/\mathrm{d}t^2$.

Using all this information, sketch the graph of $N(t)$ against t, and compare this with what you obtain by solving the equation analytically for $0 \le N \le a/b$.

11. Water flows into a cylindrical bucket of depth H and cross-sectional area A at a volume flow rate Q which is constant. There is a hole in the bottom of the bucket of cross-sectional area $a \ll A$. When the water level above the hole is h, the flow rate out of the hole is $a\sqrt{2gh}$, where g is the gravitational acceleration. Derive an equation for $\mathrm{d}h/\mathrm{d}t$. Find the equilibrium depth h_e of water, and show that it is stable.

12. In each of the following equations for $y(t)$, find the equilibrium points and classify their stability properties:

(i) $\frac{\mathrm{d}y}{\mathrm{d}t} = y(y-1)(y-2)$,

(ii) $\frac{\mathrm{d}y}{\mathrm{d}t} = -2\tan^{-1}[y/(1+y^2)]$,

*(iii) $\frac{\mathrm{d}y}{\mathrm{d}t} = y^3(\mathrm{e}^y-1)^2$.

13. Investigate the stability of the constant solutions $(u_{n+1} = u_n)$ of the discrete equation

$$u_{n+1} = 4u_n(1-u_n).$$

In the case $0 \le u_0 \le 1$, use the substitution $u_0 = \sin^2\theta$ to find the general solution and verify your stability results. Can you find an explicit form of the general solution in the case $u_0 > 1$?

*14. Two identical snowploughs plough the same stretch of road in the same direction. The first starts at $t = 0$ when the depth of snow is h_0 and the second starts from the same point T seconds later. Snow falls so that the depth of snow increases at a constant rate of k ms^{-1}. The speed of each snowplough is $k/(ah)$ where h is the depth of snow it is ploughing and a is a constant, and each snowplough clears all the snow. Show that the time taken for the first snowplough to travel x metres is

$$(\mathrm{e}^{ax}-1)h_0k^{-1} \text{ seconds.}$$

Show also that the time t by which the second snowplough has travelled x metres satisfies the equation

$$\frac{1}{a}\frac{\mathrm{d}t}{\mathrm{d}x} = t - (\mathrm{e}^{ax} - 1)h_0 k^{-1} \ .$$

Hence show that the snowploughs will collide when they have moved a distance $kT/(ah_0)$ metres.

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Examples Sheet 3

The starred questions are intended as extras: do them if you have time, but not at the expense of questions on later sheets

1. Find the general solutions of
(i) $y'' + 5y' + 6y = e^{3x}$,
(ii) $y'' + 9y = \cos 3x$,
(iii) $y'' + 2y' + 5y = 0$,
(iv) $y'' - 2y' + y = (x-1)e^x$.

2. The function $y(x)$ satisfies the linear equation

$$y'' + p(x)y' + q(x)y = 0.$$

The Wronskian $W(x)$ of two independent solutions, denoted $y_1(x)$ and $y_2(x)$, is defined to be

$$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}.$$

Let $y_1(x)$ be given. Use the Wronskian to determine a first-order inhomogeneous differential equation for $y_2(x)$. Hence, show that

$$y_2(x) = y_1(x) \int_{x_0}^{x} \frac{W(t)}{y_1(t)^2}\, dt. \qquad (*)$$

Show that $W(x)$ satisfies

$$\frac{dW}{dx} + p(x)W = 0.$$

Verify that $y_1(x) = 1 - x$ is a solution of

$$xy'' - (1 - x^2)y' - (1 + x)y = 0. \qquad (\dagger)$$

Hence, using (*) with $x_0 = 0$ and expanding the integrand in powers of t to order t^3, find the first three non-zero terms in the power series expansion for a solution, y_2, of (†) that is independent of y_1 and satisfies $y_2(0) = 0$, $y_2''(0) = 1$.

3. Find the general solutions of
(i) $y_{n+2} + y_{n+1} - 6y_n = n^2$,
(ii) $y_{n+2} - 3y_{n+1} + 2y_n = n$,
(iii) $y_{n+2} - 4y_{n+1} + 4y_n = a^n$, $\quad a$ real.

4. (i) Find the solution of $y'' - y' - 2y = 0$ that satisfies $y(0) = 1$ and is bounded as $x \to \infty$.
(ii) Solve the related difference equation

$$(y_{n+1} - 2y_n + y_{n-1}) - \tfrac{1}{2}h(y_{n+1} - y_{n-1}) - 2h^2 y_n = 0 \,,$$

and show that if $0 < h \ll 1$ the solution that satisfies $y_0 = 1$ and for which y_n is bounded as $n \to \infty$ is approximately $y_n = (1 - h + \frac{1}{2}h^2)^n$. Explain the relation with the solution of (i).

5. Show that

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dT}{dr}\right) \equiv \frac{1}{r}\frac{d^2}{dr^2}(rT)$$

and hence solve the equation

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dT}{dr}\right) = k^2 T$$

subject to the conditions that T is finite at $r = 0$ and $T(1) = 1$.

6. Given the solution $y_1(x)$, find a second solution of the following equations:
(i) $x(x+1)y'' + (x-1)y' - y = 0, \quad y_1(x) = (x+1)^{-1}$;
(ii) $xy'' - y' - 4x^3 y = 0, \quad y_1(x) = e^{x^2}$.

*7. The n functions $y_j(x)$ $(1 \le j \le n)$ are independent solutions of the equation

$$y^{(n)}(x) + p_1(x)y^{(n-1)}(x) + \cdots + p_{n-1}(x)y'(x) + p_n(x)y(x) = 0.$$

Let $\mathbf{W}$ be the $n \times n$ matrix whose i,j element W_{ij} is $y_j^{(i-1)}(x)$ (so that $\det\mathbf{W} = \mathcal{W}$, the Wronskian). Find a matrix $\mathbf{A}$, which does not explicitly involve the y_j such that

$$\mathbf{W}' = \mathbf{A}\,\mathbf{W}$$

where $\mathbf{W}'$ is the matrix whose elements are given by $(\mathbf{W}')_{ij} = W'_{ij}$. Using the identity

$$(\ln\det\mathbf{W})' = \operatorname{trace}\left(\mathbf{W}'\mathbf{W}^{-1}\right),$$

express $\mathcal{W}$ in terms of $p_1(x)$. [You can prove this identity by writing $\mathbf{W} = \mathbf{PDP}^{-1}$ where $\mathbf{D}$ is in Jordan normal form (which is upper triangular) and using trace $\mathbf{ABC}$ = trace $\mathbf{BCA}$.]

8. Let $y(x)$ satisfy the inhomogeneous equation

$$y'' - 2x^{-1}y' + 2x^{-2}y = f(x)\,. \qquad (*)$$

Set

$$\begin{pmatrix} y \\ y' \end{pmatrix} = u(x)\begin{pmatrix} y_1 \\ y_1' \end{pmatrix} + v(x)\begin{pmatrix} y_2 \\ y_2' \end{pmatrix},$$

where $y_1(x)$ and $y_2(x)$ are two independent solutions of $(*)$ when $f(x) = 0$, and $u(x)$ and $v(x)$ are unknown functions. Obtain first-order differential equations for $u(x)$ and $v(x)$, and hence find the most general solution of (∗) in the case $f(x) = x \sin x$. Are the functions $u(x)$ and $v(x)$ completely determined by this procedure?

9. A large oil tanker of mass W floats on the sea of density ρ. Suppose the tanker is given a small downward displacement z. The upward force is equal to the weight of water displaced (Archimedes' Principle). If the cross-sectional area A of the tanker at the water surface is constant, show that this upward force is $g\rho Az$, and hence that

$$\ddot{z} + \frac{g\rho A}{W} z = 0 \ .$$

Suppose now that a mouse exercises on the deck of the tanker producing a vertical force $m \sin \omega t$, where $\omega = (g\rho A/W)^{1/2}$. Show that the tanker will eventually sink. In practice, as the vertical motion of the tanker increases, waves will be generated. Suppose they produce an additional damping $2k\dot{z}$. Discuss the motion for a range of values of k.

10. Find and sketch the solution of

$$\ddot{y} + y = H(t - \pi) - H(t - 2\pi),$$

where H is the Heaviside step function, subject to

$$y(0) = \dot{y}(0) = 0 \ ,$$

$$y(t) \ \text{ and } \ \dot{y}(t) \ \text{ continuous at } t = \pi, 2\pi \ .$$

11. Solve

$$y'' - 4y = \delta(x - a),$$

where δ is the Dirac delta function, subject to the boundary conditions that y is bounded as $|x| \to \infty$. Sketch the solution.

12. Solve

$$\ddot{y} + 2\dot{y} + 5y = 2\delta(t),$$

where δ is the Dirac delta function, given that $y = 0$ for $t < 0$. Give an example of a physical system that this describes.

*13. Show that, for suitably chosen $P(x)$, the transformation $y(x) = P(x)v(x)$ reduces the equation

$$y'' + p(x)y' + q(x)y = 0$$

to the form

$$v'' + J(x)v = 0\ . \qquad (\dagger)$$

The sequence of functions $v_n(x)$ is defined, for a given function $J(t)$ and in a given range $0 \le x \le R$, by $v_0(x) = a + bx$ and

$$v_n(x) = \int_0^x (t-x)J(t)v_{n-1}(t)\mathrm{d}t. \quad (n \ge 1)\ .$$

Show that $v_n''(x) + J(x)v_{n-1} = 0$ $(n \ge 1)$ and deduce that $v(x) = \sum_0^\infty v_n(x)$ satisfies $(\dagger)$ with the initial conditions $v(0) = a,\ v'(0) = b$.
[N.B. You may assume that the sum which defines $v(x)$ converges sufficiently nicely to allow term-by-term differentiation. In fact, you can show by induction that if $|J(x)| < m$ and $|v_0(x)| < A$ for the range of x under consideration, then $|v_n(x)| \le Am^n x^{2n}/(2n)!$ – try it! Convergence is therefore exponentially fast.]

What does this tell us about the existence problem for general second-order linear equations with given initial conditions?

*14 *The expanding universe.* Einstein's equations for a flat isotropic and homogeneous universe can be written as :

$$\frac{\ddot{a}}{a} = -\frac{4\pi}{3}(\rho + 3p) + \frac{\Lambda}{3}, \qquad H \equiv \left(\frac{\dot{a}}{a}\right) = \left(\frac{8\pi}{3}\rho + \frac{\Lambda}{3}\right)^{1/2},$$

where a is the scale factor measuring the expansion of the universe ($\dot{a} > 0$), ρ and p are the time-dependent energy density and pressure of matter, Λ is the cosmological constant and $H > 0$ the Hubble parameter. Use these equations to establish the following: If $\Lambda \sim 0$ and $\rho + 3p > 0$ the acceleration $\ddot{a} < 0$ and the graph of $a(t)$ must be concave downward implying that at a finite time a must reach $a = 0$ (the big bang). Using the tangent of the graph at present time $t = t_0$ show that the age of the universe is bounded by $t_0 < H^{-1}(t_0)$.
Consider the physical situations of a matter dominated universe ($\Lambda, p \sim 0$) and a radiation dominated universe ($\Lambda \sim 0, p = \rho/3$). In each case, reduce the two equations above to one single differential equation for a which is homogeneous in t (invariant under $t \to \lambda t$) and then show that there is a solution of the type $a = t^\alpha$. Determine the value of α for each case and verify that $\ddot{a} < 0$.
Now consider a Λ dominated universe ($\rho, p << \Lambda$), solve the differential equation for $a(t)$ and show that it corresponds to an accelerated universe ($\ddot{a} > 0$)for $\Lambda > 0$. This could describe the universe today and/or a very early period of exponential expansion known as inflation.

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Examples Sheet 4

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1. By finding solutions as power series in x solve

$$4xy'' + 2(1-x)y' - y = 0\ .$$

2. Find the two independent series solutions about $x = 0$ of

$$y'' - 2xy' + \lambda y = 0,$$

for a constant λ. Show that for $\lambda = 2n$, with n a positive integer, the solutions are polynomials of degree n. These are the Hermite polynomials relevant for the solution of the simple harmonic oscillator in quantum mechanics.

3. What is the nature of the point $x = 0$ with respect to the differential equation

$$x^2y'' - xy' + (1 - \gamma x)y = 0.$$

Find a series solution about $x = 0$ for $\gamma \neq 0$ and write down the form of a second, independent solution. Find two independent solutions of the equation for $\gamma = 0$.

4. Bessel's equation is

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0\ .$$

For $\nu = 0$, find a solution in the form of a power series about $x = 0$.

For $\nu = \frac{1}{2}$, find two independent series solutions of this form. Perform also the change of variables $y(x) = z(x)/\sqrt{x}$ to simplify the equation, solve for $z(x)$ and compare with the series result.

5. Find the positions and nature of each of the stationary points of

$$f(x, y) = x^3 + 3xy^2 - 3x$$

and draw a rough sketch of the contours of f.

6. Find the positions of each of the stationary points of

$$f(x, y) = \sin\left(\frac{x-y}{2}\right)\sin y$$

in $0 < x < 2\pi, 0 < y < 2\pi$. By using this information and identifying the zero contours of f, sketch the contours of f and identify the nature of the stationary points.

7. For the function $f(x, y, z) = x^2 + y^2 + z^3 - 3z$, find ∇f.

   (i) What is the rate of change of $f(x, y, z)$ in the direction normal to the cylinder $x^2 + y^2 = 25$ at the point $(3, -4, 4)$?

   (ii) At which points does ∇f have no component in the z direction?

   (iii) Find and classify the stationary points of f.

   (iv) Sketch the contours of f and add to the sketch a few arrows showing the directions of ∇f.

8. Use matrix methods to solve

$$y' = y - 3z - 6e^x \ , \quad z' = y + 5z$$

for $y(x),\ z(x)$ subject to initial conditions $y(0) = 1, z(0) = 0$.

9. Consider the linear system

$$\dot{\mathbf{x}}(t) + \mathbf{P}\ \mathbf{x}(t) = \mathbf{z}(t) \ ,$$

where $\mathbf{x}(t), \mathbf{z}(t)$ are 2-vectors, $\mathbf{P}$ is a real constant 2×2 matrix and $\mathbf{z}(t)$ is a given input. Show that free motion (i.e. $\mathbf{z}(t) = 0$) is purely oscillatory (i.e. no growth or decay) if and only if trace $\mathbf{P} = 0$ and $\det \mathbf{P} > 0$. [*The trace of a square matrix is the sum of its diagonal elements.*]

Consider

$$\dot{x} + x - y = \cos 2t, \quad \dot{y} + 5x - y = \cos 2t + 2a \sin 2t \ ,$$

for various values of the real constant a. For what value(s) of a is there resonance? What general principle does this illustrate?

10. Show that the system

$$\dot{x} = e^{x+y} - y,$$
$$\dot{y} = -x + xy$$

has only one fixed point. Find the linearized system about this point and discuss its stability. Draw the phase portrait near the fixed point.

11. Use matrix methods to find the general solution of the equations

$$\dot{x} = 3x + 2y \ , \quad \dot{y} = -5x - 3y \ . \qquad (\dagger)$$

Sketch the phase-plane trajectories in the vicinity of the origin.

*Show that the set of equations $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$, where $\mathbf{x}(t)$ is a column vector and $\mathbf{A}$ is an $n \times n$ matrix with constant elements, has solutions of the form $\mathbf{x} = \exp(\mathbf{A}t)\mathbf{x}_0$, where $\mathbf{x}_0$ is a constant vector and

$$\exp(\mathbf{A}t) \equiv \mathbf{I} + \mathbf{A}t + \frac{1}{2!}\mathbf{A}^2t^2 + \frac{1}{3!}\mathbf{A}^3t^3 + \ldots \ .$$

Use this method to solve equations (†).

Would you expect this method to work if the elements of $\mathbf{A}$ are not constant?

12. Carnivorous hunters of population h prey on vegetarians of population p. In the absence of hunters the prey will increase in number until their population is limited by the availability of food. In the absence of prey the hunters will eventually die out. The equations governing the evolution of the populations are

$$\dot{p} = p(1-p) - ph, \qquad \dot{h} = \frac{h}{8}\left(\frac{p}{b} - 1\right), \qquad (*)$$

where b is a positive constant, and $h(t)$ and $p(t)$ are non-negative functions of time t.

In the two cases $0 < b < 1/2$ and $b > 1$ determine the location and the stability properties of the critical points of (*). In both of these cases sketch the typical solution trajectories and briefly describe the ultimate fate of hunters and prey.

13. Consider the change of variables

$$x = e^{-s}\sin t, \quad y = e^{-s}\cos t \quad \text{such that} \quad u(x,y) = v(s,t).$$

(i) Use the chain rule to express $\partial v/\partial s$ and $\partial v/\partial t$ in terms of $x, y, \partial u/\partial x$ and $\partial u/\partial y$.

(ii) Find, similarly, an expression for $\partial^2 v/\partial t^2$.

(iii) Hence transform the equation

$$y^2\frac{\partial^2 u}{\partial x^2} - 2xy\frac{\partial^2 u}{\partial x \partial y} + x^2\frac{\partial^2 u}{\partial y^2} = 0$$

into a partial differential equation for v.

14. Solve

$$\frac{\partial y}{\partial t} - 2\frac{\partial y}{\partial x} + y = 0$$

for $y(x,t)$ given $\quad y(x,0) = e^{x^2}$.

[*Hint: consider paths in the $x - t$ plane with $x = x_0 - 2t$ (x_0 constant).*]

15. The function $\theta(x,t)$ obeys the diffusion equation

$$\frac{\partial \theta}{\partial t} = \frac{\partial^2 \theta}{\partial x^2}.$$

Find, by substitution, solutions of the form

$$\theta(x,t) = f(t) \exp[-(x+a)^2/4(t+b)],$$

where a and b are arbitrary constants and the function f is to be determined.
Hence find a solution which satisfies the initial condition

$$\theta(x,0) = \exp[-(x-2)^2] - \exp[-(x+2)^2]$$

and sketch its behaviour for $t \geq 0$.

16. Solve the partial differential equation

$$\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad (*)$$

for $u(x,y)$ by making a change of variables as follows. Define new variables

$$\xi = x - y, \quad \eta = x,$$

and evaluate the partial derivatives of x and y with respect to ξ and η. Writing $v(\xi,\eta) = u(x,y)$, use these derivatives and the chain rule to show that

$$\frac{\partial v}{\partial \eta} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y},$$

and that the equation

$$\frac{\partial^2 v}{\partial \eta^2} = 0$$

is equivalent to equation $(*)$.

Deduce that the most general solution of (*) is

$$u(x,y) = f(x-y) + xg(x-y),$$

where f and g are arbitrary functions.

Solve $(*)$ completely given that $u(0,y) = 0$ for all y, whilst $u(x,1) = x^2$ for all x.

Comments and corrections may be sent by email to mgw1@cam.ac.uk