Part IA — Differential Equations
Definitions

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Basic calculus
Informal treatment of differentiation as a limit, the chain rule, Leibnitz's rule, Taylor series, informal treatment of O and o notation and l'Hôpital's rule; integration as an area, fundamental theorem of calculus, integration by substitution and parts. [3]

Informal treatment of partial derivatives, geometrical interpretation, statement (only) of symmetry of mixed partial derivatives, chain rule, implicit differentiation. Informal treatment of differentials, including exact differentials. Differentiation of an integral with respect to a parameter. [2]

First-order linear differential equations
Equations with constant coefficients: exponential growth, comparison with discrete equations, series solution; modelling examples including radioactive decay.

Equations with non-constant coefficients: solution by integrating factor. [2]

Nonlinear first-order equations
Separable equations. Exact equations. Sketching solution trajectories. Equilibrium solutions, stability by perturbation; examples, including logistic equation and chemical kinetics. Discrete equations: equilibrium solutions, stability; examples including the logistic map. [4]

Higher-order linear differential equations
Complementary function and particular integral, linear independence, Wronskian (for second-order equations), Abel's theorem. Equations with constant coefficients and examples including radioactive sequences, comparison in simple cases with difference equations, reduction of order, resonance, transients, damping. Homogeneous equations. Response to step and impulse function inputs; introduction to the notions of the Heaviside step-function and the Dirac delta-function. Series solutions including statement only of the need for the logarithmic solution. [8]

Multivariate functions: applications
Directional derivatives and the gradient vector. Statement of Taylor series for functions on $\mathbb{R}^n$. Local extrema of real functions, classification using the Hessian matrix. Coupled first order systems: equivalence to single higher order equations; solution by matrix methods. Non-degenerate phase portraits local to equilibrium points; stability.

Simple examples of first- and second-order partial differential equations, solution of the wave equation in the form $f(x+ct)+g(x-ct)$. [5]

Contents

0 Introduction

1 Differentiation

1.1 Differentiation

Definition (Derivative of function)**.** The *derivative* of a function $f(x)$ with respect to x, interpreted as the rate of change of $f(x)$ with x, is

$$\frac{\mathrm{d}f}{\mathrm{d}x} = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}.$$

A function $f(x)$ is differentiable at x if the limit exists (i.e. the left-hand and right-hand limits are equal).

Notation. We write $\frac{\mathrm{d}f}{\mathrm{d}x} = f'(x) = \frac{\mathrm{d}}{\mathrm{d}x}f(x)$. Also, $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}}{\mathrm{d}x}f(x)\right) = \frac{\mathrm{d}^2}{\mathrm{d}x^2}f(x) = f''(x)$.

Note that the notation f' represents the derivative with respect to the argument. For example, $f'(2x) = \frac{\mathrm{d}f}{\mathrm{d}(2x)}$

1.2 Small o and big O notations

Definition (O and o notations)**.**

(i) "$f(x) = o(g(x))$ as $x \to x_0$" if $\lim\limits_{x\to x_0}\frac{f(x)}{g(x)} = 0$. Intuitively, $f(x)$ is much smaller than $g(x)$.

(ii) "$f(x) = O(g(x))$ as $x \to x_0$" if $\frac{f(x)}{g(x)}$ is bounded as $x \to x_0$. Intuitively, $f(x)$ is about as big as $g(x)$.

Note that for $f(x) = O(g(x))$ to be true, $\lim\limits_{x\to x_0}\dfrac{f(x)}{g(x)}$ need not exist.

Usually, x_0 is either 0 or infinity. Clearly, we have $f(x) = o(g(x))$ implies $f(x) = O(g(x))$.

1.3 Methods of differentiation

1.4 Taylor's theorem

1.5 L'Hopital's rule

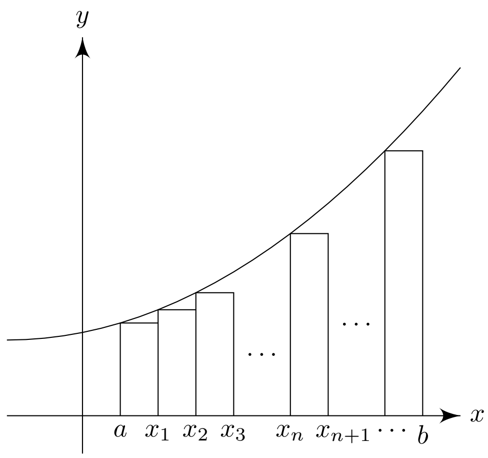
2 Integration

2.1 Integration

Definition (Integral)**.** An *integral* is the limit of a sum, e.g.

$$\int_a^b f(x)\ \mathrm{d}x = \lim_{\Delta x \to 0} \sum_{n=0}^{N} f(x_n)\Delta x.$$

For example, we can take $\Delta x = \frac{b-a}{N}$ and $x_n = a + n\Delta x$. Note that an integral need not be defined with this particular Δx and x_n. The term "integral" simply refers to any limit of a sum (The usual integrals we use are a special kind known as Riemann integral, which we will study formally in Analysis I). Pictorially, we have



Notation. We write $\int f(x)\ \mathrm{d}x = \int^x f(t)\ \mathrm{d}t$, where the unspecified lower limit gives rise to the constant of integration.

2.2 Methods of integration

3 Partial differentiation

3.1 Partial differentiation

Definition (Partial derivative)**.** Given a function of several variables $f(x, y)$, the *partial derivative* of f with respect to x is the rate of change of f as x varies, keeping y constant. It is given by

$$\left.\frac{\partial f}{\partial x}\right|_y = \lim_{\delta x \to 0} \frac{f(x + \delta x, y) - f(x, y)}{\delta x}$$

Notation.

$$f_x = \frac{\partial f}{\partial x}, \quad f_{xy} = \frac{\partial^2 f}{\partial y \partial x}.$$

3.2 Chain rule

3.3 Implicit differentiation

3.4 Differentiation of an integral wrt parameter in the integrand

4 First-order differential equations

4.1 The exponential function

Definition (Exponential function). $\exp(x) = e^x$ is the unique function f satisfying $f'(x) = f(x)$ and $f(0) = 1$.

We write the inverse function as $\ln x$ or $\log x$.

Definition (Eigenfunction). An *eigenfunction* under the differential operator is a function whose functional form is unchanged by the operator. Only its magnitude is changed. i.e.

$$\frac{\mathrm{d}f}{\mathrm{d}x} = \lambda f$$

4.2 Homogeneous linear ordinary differential equations

Definition (Linear differential equation). A differential equation is *linear* if the dependent variable (y, y', y'' etc.) appears only linearly.

Definition (Homogeneous differential equation). A differential equation is *homogeneous* if $y = 0$ is a solution.

Definition (Differential equation with constant coefficients). A differential equation has *constant coefficients* if the independent variable x does not appear explicitly.

Definition (First-order differential equation). A differential equation is *first-order* if only first derivatives are involved.

4.3 Forced (inhomogeneous) equations

4.3.1 Constant forcing

4.3.2 Eigenfunction forcing

4.4 Non-constant coefficients

4.5 Non-linear equations

4.5.1 Separable equations

Definition (Separable equation). A first-order differential equation is *separable* if it can be manipulated into the following form:

$$q(y)\ \mathrm{d}y = p(x)\ \mathrm{d}x.$$

in which case the solution can be found by integration

$$\int q(y)\ \mathrm{d}y = \int p(x)\ \mathrm{d}x.$$

4.5.2 Exact equations

Definition (Exact equation)**.** $Q(x,y)\frac{\mathrm{d}y}{\mathrm{d}x} + P(x,y) = 0$ is an *exact equation* iff the differential form $Q(x,y)\ \mathrm{d}y + P(x,y)\ \mathrm{d}x$ is *exact*, i.e. there exists a function $f(x,y)$ for which

$$\mathrm{d}f = Q(x,y)\ \mathrm{d}y + P(x,y)\ \mathrm{d}x$$

Definition (Simply-connected domain)**.** A domain $\mathcal{D}$ is simply-connected if it is connected and any closed curve in $\mathcal{D}$ can be shrunk to a point in $\mathcal{D}$ without leaving $\mathcal{D}$.

4.6 Solution curves (trajectories)

4.7 Fixed (equilibrium) points and stability

Definition (Equilibrium/fixed point)**.** An *equilibrium point* or a *fixed point* of a differential equation is a constant solution $y = c$. This corresponds to $\frac{\mathrm{d}y}{\mathrm{d}t} = 0$ for all t.

Definition (Stability of fixed point)**.** An equilibrium is *stable* if whenever y is deviated slightly from the constant solution $y = c$, $y \to c$ as $t \to \infty$. An equilibrium is *unstable* if the deviation grows as $t \to \infty$.

4.7.1 Perturbation analysis

4.7.2 Autonomous systems

Definition (Autonomous system)**.** An *autonomous system* is a system in the form $\dot{y} = f(y)$, where the derivative is only (explicitly) dependent on y.

4.7.3 Logistic Equation

4.8 Discrete equations (Difference equations)

5 Second-order differential equations

5.1 Constant coefficients

5.1.1 Complementary functions

Definition (Characteristic equation)**.** The *characteristic equation* of a (second-order) differential equation $ay'' + by' + c = 0$ is

$$a\lambda^2 + b\lambda + c = 0.$$

5.1.2 Second complementary function

5.1.3 Phase space

Definition (Wronskian)**.** Given a differential equation with solutions y_1, y_2, the *Wronskian* is the determinant

$$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}.$$

Definition (Independent solutions)**.** Two solutions $y_1(x)$ and $y_2(x)$ are *independent* solutions of the differential equation if and only if $\mathbf{Y}_1$ and $\mathbf{Y}_2$ are linearly independent as vectors in the phase space for some x, i.e. iff the Wronskian is non-zero for some x.

5.2 Particular integrals

5.2.1 Guessing

5.2.2 Resonance

5.2.3 Variation of parameters

5.3 Linear equidimensional equations

Definition (Equidimensional equation)**.** An equation is *equidimensional* if it has the form

$$ax^2y'' + bxy' + cy = f(x),$$

where a, b, c are constants.

5.4 Difference equations

5.5 Transients and damping

5.6 Impulses and point forces

5.6.1 Dirac delta function

Definition (Dirac delta function)**.** The *Dirac delta function* is defined by

$$\delta(x) = \lim_{\varepsilon \to 0} D(x; \varepsilon)$$

on the understanding that we can only use its integral properties. For example, when we write

$$\int_{-\infty}^{\infty} g(x)\delta(x)\ \mathrm{d}x,$$

we actually mean

$$\lim_{\varepsilon\to 0}\int_{-\infty}^{\infty} g(x)D(x;\varepsilon)\ \mathrm{d}x.$$

In fact, this is equal to $g(0)$.

More generally, $\int_a^b g(x)\delta(x-c)\ \mathrm{d}x = g(c)$ if $c \in (a,b)$ and 0 otherwise, provided g is continuous at $x = c$.

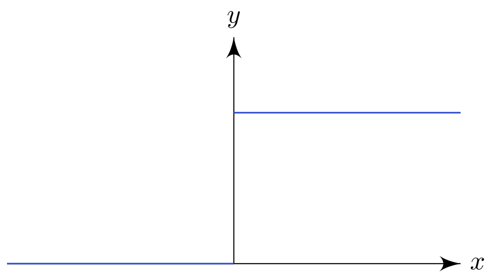
5.7 Heaviside step function

Definition (Heaviside step function)**.** Define the Heaviside step function as:

$$H(x) = \int_{-\infty}^{x} \delta(t)\ \mathrm{d}t$$

We have

$$H(x) = \begin{cases} 0 & x < 0 \\ 1 & x > 0 \\ \text{undefined} & x = 0 \end{cases}$$



By the fundamental theorem of calculus,

$$\frac{\mathrm{d}H}{\mathrm{d}x} = \delta(x)$$

But remember that these functions and relationships can only be used inside integrals.

6 Series solutions

Definition (Ordinary and singular points)**.** The point $x = x_0$ is an *ordinary point* of the differential equation if $\frac{q}{p}$ and $\frac{r}{p}$ have Taylor series about x_0 (i.e. are "analytic", cf. Complex Analysis). Otherwise, x_0 is a *singular point*.

If x_0 is a singular point but the equation can be written as

$$P(x)(x - x_0)^2 y'' + Q(x)(x - x_0)y' + R(x)y = 0,$$

where $\frac{Q}{P}$ and $\frac{R}{P}$ have Taylor series about x_0, then x_0 is a *regular singular point*.

7 Directional derivative

7.1 Gradient vector

Definition (Directional derivative)**.** The *directional derivative* of f in the direction of $\hat{\mathbf{s}}$ is

$$\frac{\mathrm{d}f}{\mathrm{d}s} = \hat{\mathbf{s}} \cdot \nabla f.$$

Definition (Gradient vector)**.** The *gradient vector* ∇f is defined as the vector that satisfies

$$\frac{\mathrm{d}f}{\mathrm{d}s} = \hat{\mathbf{s}} \cdot \nabla f.$$

7.2 Stationary points

7.3 Taylor series for multi-variable functions

Definition (Hessian matrix)**.** The *Hessian matrix* is the matrix

$$\nabla\nabla f = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}$$

7.4 Classification of stationary points

Definition (Signature of Hessian matrix)**.** The *signature* of H is the pattern of the signs of the subdeterminants:

$$\underbrace{f_{xx}}_{|H_1|}, \underbrace{\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}}_{|H_2|}, \cdots, \underbrace{\begin{vmatrix} f_{xx} & f_{xy} & \cdots & f_{xz} \\ f_{yx} & f_{yy} & \cdots & f_{yz} \\ \vdots & \vdots & \ddots & \vdots \\ f_{zx} & f_{zy} & \cdots & f_{zz} \end{vmatrix}}_{|H_n| = |H|}$$

7.5 Contours of $f(x, y)$

8 Systems of differential equations

8.1 Linear equations

8.2 Nonlinear dynamical systems

Definition (Equilibrium point)**.** An *equilibrium point* is a point in which $\dot{x} = \dot{y} = 0$ at $\mathbf{x}_0 = (x_0, y_0)$.

9 Partial differential equations (PDEs)

9.1 First-order wave equation

9.2 Second-order wave equation

9.3 The diffusion equation