Part IA — Vector Calculus
Theorems with proof

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Curves in $\mathbb{R}^3$
Parameterised curves and arc length, tangents and normals to curves in $\mathbb{R}^3$, the radius of curvature. [1]

Integration in $\mathbb{R}^2$ and $\mathbb{R}^3$
Line integrals. Surface and volume integrals: definitions, examples using Cartesian, cylindrical and spherical coordinates; change of variables. [4]

Vector operators
Directional derivatives. The gradient of a real-valued function: definition; interpretation as normal to level surfaces; examples including the use of cylindrical, spherical *and general orthogonal curvilinear* coordinates.

Divergence, curl and ∇^2 in Cartesian coordinates, examples; formulae for these operators (statement only) in cylindrical, spherical *and general orthogonal curvilinear* coordinates. Solenoidal fields, irrotational fields and conservative fields; scalar potentials. Vector derivative identities. [5]

Integration theorems
Divergence theorem, Green's theorem, Stokes's theorem, Green's second theorem: statements; informal proofs; examples; application to fluid dynamics, and to electromagnetism including statement of Maxwell's equations. [5]

Laplace's equation
Laplace's equation in $\mathbb{R}^2$ and $\mathbb{R}^3$: uniqueness theorem and maximum principle. Solution of Poisson's equation by Gauss's method (for spherical and cylindrical symmetry) and as an integral. [4]

Cartesian tensors in $\mathbb{R}^3$
Tensor transformation laws, addition, multiplication, contraction, with emphasis on tensors of second rank. Isotropic second and third rank tensors. Symmetric and antisymmetric tensors. Revision of principal axes and diagonalization. Quotient theorem. Examples including inertia and conductivity. [5]

Contents

0 Introduction

1 Derivatives and coordinates

1.1 Derivative of functions

Proposition.
$$\mathbf{F}'(x) = F_i'(x)\mathbf{e}_i.$$

Proposition.

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(f\mathbf{g}) &= \frac{\mathrm{d}f}{\mathrm{d}t}\mathbf{g} + f\frac{\mathrm{d}\mathbf{g}}{\mathrm{d}t}\\ \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{g}\cdot\mathbf{h}) &= \frac{\mathrm{d}\mathbf{g}}{\mathrm{d}t}\cdot\mathbf{h} + \mathbf{g}\cdot\frac{\mathrm{d}\mathbf{h}}{\mathrm{d}t}\\ \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{g}\times\mathbf{h}) &= \frac{\mathrm{d}\mathbf{g}}{\mathrm{d}t}\times\mathbf{h} + \mathbf{g}\times\frac{\mathrm{d}\mathbf{h}}{\mathrm{d}t}\end{aligned}$$

Note that the order of multiplication must be retained in the case of the cross product.

Theorem. The gradient is
$$\nabla f = \frac{\partial f}{\partial x_i}\mathbf{e}_i$$

Theorem (Chain rule)**.** Given a function $f(\mathbf{r}(u))$,
$$\frac{\mathrm{d}f}{\mathrm{d}u} = \nabla f\cdot\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}u} = \frac{\partial f}{\partial x_i}\frac{\mathrm{d}x_i}{\mathrm{d}u}.$$

Theorem. The derivative of $\mathbf{F}$ is given by
$$M_{ji} = \frac{\partial y_j}{\partial x_i}.$$

Theorem (Chain rule)**.** Suppose $g:\mathbb{R}^p\to\mathbb{R}^n$ and $f:\mathbb{R}^n\to\mathbb{R}^m$. Suppose that the coordinates of the vectors in $\mathbb{R}^p, \mathbb{R}^n$ and $\mathbb{R}^m$ are u_a, x_i and y_r respectively. By the chain rule,
$$\frac{\partial y_r}{\partial u_a} = \frac{\partial y_r}{\partial x_i}\frac{\partial x_i}{\partial u_a},$$
with summation implied. Writing in matrix form,
$$M(f\circ g)_{ra} = M(f)_{ri}M(g)_{ia}.$$
Alternatively, in operator form,
$$\frac{\partial}{\partial u_a} = \frac{\partial x_i}{\partial u_a}\frac{\partial}{\partial x_i}.$$

1.2 Inverse functions

1.3 Coordinate systems

2 Curves and Line

2.1 Parametrised curves, lengths and arc length

Proposition. Let s denote the arclength of a curve $\mathbf{r}(u)$. Then

$$\frac{\mathrm{d}s}{\mathrm{d}u} = \pm\left|\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}u}\right| = \pm|\mathbf{r}'(u)|$$

with the sign depending on whether it is in the direction of increasing or decreasing arclength.

Proposition. $\mathrm{d}s = \pm|\mathbf{r}'(u)|\mathrm{d}u$

2.2 Line integrals of vector fields

2.3 Gradients and Differentials

Theorem. If $\mathbf{F} = \nabla f(\mathbf{r})$, then

$$\int_C \mathbf{F}\cdot \mathrm{d}\mathbf{r} = f(\mathbf{b}) - f(\mathbf{a}),$$

where $\mathbf{b}$ and $\mathbf{a}$ are the end points of the curve.

In particular, the line integral does *not* depend on the curve, but the end points only. This is the vector counterpart of the fundamental theorem of calculus. A special case is when C is a closed curve, then $\oint_C \mathbf{F}\cdot\mathrm{d}\mathbf{r} = 0$.

Proof. Let $\mathbf{r}(u)$ be any parametrization of the curve, and suppose $\mathbf{a} = \mathbf{r}(\alpha)$, $\mathbf{b} = \mathbf{r}(\beta)$. Then

$$\int_C \mathbf{F}\cdot\mathrm{d}\mathbf{r} = \int_C \nabla f\cdot\mathrm{d}\mathbf{r} = \int \nabla f\cdot\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}u}\,\mathrm{d}u.$$

So by the chain rule, this is equal to

$$\int_\alpha^\beta \frac{\mathrm{d}}{\mathrm{d}u}(f(\mathbf{r}(u)))\,\mathrm{d}u = [f(\mathbf{r}(u))]_\alpha^\beta = f(\mathbf{b}) - f(\mathbf{a}). \qquad \square$$

Proposition. If $\mathbf{F} = \nabla f$ for some f, then

$$\frac{\partial F_i}{\partial x_j} = \frac{\partial F_j}{\partial x_i}.$$

This is because both are equal to $\partial^2 f/\partial x_i\partial x_j$.

Proposition.

$$\begin{aligned}\mathrm{d}(\lambda f + \mu g) &= \lambda\mathrm{d}f + \mu\mathrm{d}g\\ \mathrm{d}(fg) &= (\mathrm{d}f)g + f(\mathrm{d}g)\end{aligned}$$

2.4 Work and potential energy

3 Integration in $\mathbb{R}^2$ and $\mathbb{R}^3$

3.1 Integrals over subsets of $\mathbb{R}^2$

Proposition.
$$\int_D f(x, y)\;\mathrm{d}A = \int_Y \left(\int_{x_y} f(x, y)\;\mathrm{d}x\right)\mathrm{d}y.$$
with x_y ranging over $\{x : (x, y) \in D\}$.

Theorem (Fubini's theorem)**.** If f is a continuous function and D is a compact (i.e. closed and bounded) subset of $\mathbb{R}^2$, then
$$\iint f\;\mathrm{d}x\;\mathrm{d}y = \iint f\;\mathrm{d}y\;\mathrm{d}x.$$

While we have rather strict conditions for this theorem, it actually holds in many more cases, but those situations have to be checked manually.

Proposition. $\mathrm{d}A = \mathrm{d}x\;\mathrm{d}y$ in Cartesian coordinates.

Proposition. Take separable $f(x, y) = h(y)g(x)$ and D be a rectangle $\{(x, y) : a \leq x \leq b, c \leq y \leq d\}$. Then
$$\int_D f(x, y)\;\mathrm{d}x\;\mathrm{d}y = \left(\int_a^b g(x)\;\mathrm{d}x\right)\left(\int_c^d h(y)\;\mathrm{d}y\right)$$

3.2 Change of variables for an integral in $\mathbb{R}^2$

Proposition. Suppose we have a change of variables $(x, y) \leftrightarrow (u, v)$ that is smooth and invertible, with regions D, D' in one-to-one correspondence. Then
$$\int_D f(x, y)\;\mathrm{d}x\;\mathrm{d}y = \int_{D'} f(x(u, v), y(u, v))|J|\;\mathrm{d}u\;\mathrm{d}v,$$
where
$$J = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$
is the Jacobian. In other words,
$$\mathrm{d}x\;\mathrm{d}y = |J|\;\mathrm{d}u\;\mathrm{d}v.$$

Proof. Since we are writing $(x(u, v), y(u, v))$, we are actually transforming from (u, v) to (x, y) and not the other way round.

Suppose we start with an area $\delta A' = \delta u \delta v$ in the (u, v) plane. Then by Taylors' theorem, we have
$$\delta x = x(u + \delta u, v + \delta v) - x(u, v) \approx \frac{\partial x}{\partial u}\delta u + \frac{\partial x}{\partial v}\delta v.$$
We have a similar expression for δy and we obtain
$$\begin{pmatrix} \delta x \\ \delta y \end{pmatrix} \approx \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}\begin{pmatrix} \delta u \\ \delta v \end{pmatrix}$$

Recall from Vectors and Matrices that the determinant of the matrix is how much it scales up an area. So the area formed by δx and δy is $|J|$ times the area formed by δu and δv. Hence

$$\mathrm{d}x\ \mathrm{d}y = |J|\ \mathrm{d}u\ \mathrm{d}v. \qquad \square$$

3.3 Generalization to $\mathbb{R}^3$

Proposition. $\mathrm{d}V = \mathrm{d}x\ \mathrm{d}y\ \mathrm{d}z$.

Proposition.

$$\int_V f\ \mathrm{d}x\ \mathrm{d}y\ \mathrm{d}z = \int_V f|J|\ \mathrm{d}u\ \mathrm{d}v\ \mathrm{d}w,$$

with

$$J = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

Proposition. In cylindrical coordinates,

$$\mathrm{d}V = \rho\ \mathrm{d}\rho\ \mathrm{d}\varphi\ \mathrm{d}z.$$

In spherical coordinates

$$\mathrm{d}V = r^2 \sin\theta\ \mathrm{d}r\ \mathrm{d}\theta\ \mathrm{d}\varphi.$$

Proof. Loads of algebra. $\square$

3.4 Further generalizations

Proposition.

$$\int_D f(x_1, x_2, \cdots x_n)\ \mathrm{d}x_1\ \mathrm{d}x_2 \cdots\ \mathrm{d}x_n = \int_{D'} f(\{x_i(\mathbf{u})\})|J|\ \mathrm{d}u_1\ \mathrm{d}u_2 \cdots\ \mathrm{d}u_n.$$

4 Surfaces and surface integrals

4.1 Surfaces and Normal

Proposition. ∇f is the normal to the surface $f(\mathbf{r}) = c$.

4.2 Parametrized surfaces and area

Proposition. The *vector area element* is

$$\mathrm{d}\mathbf{S} = \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \, \mathrm{d}u \, \mathrm{d}v.$$

The *scalar area element* is

$$\mathrm{d}S = \left| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right| \, \mathrm{d}u \, \mathrm{d}v.$$

4.3 Surface integral of vector fields

4.4 Change of variables in $\mathbb{R}^2$ and $\mathbb{R}^3$ revisited

5 Geometry of curves and surfaces

Theorem (Theorema Egregium)**.** K is *intrinsic* to the surface S. It can be expressed in terms of lengths, angles etc. which are measured entirely on the surface. So K can be defined on an arbitrary surface without embedding it on a higher dimension surface.

Theorem (Gauss-Bonnet theorem)**.**

$$\theta_1 + \theta_2 + \theta_3 = \pi + \int_D K \, \mathrm{d}A,$$

integrating over the area of the triangle.

6 Div, Grad, Curl and ∇

6.1 Div, Grad, Curl and ∇

Proposition. Let f, g be scalar functions, $\mathbf{F}, \mathbf{G}$ be vector functions, and μ, λ be constants. Then

$$\begin{aligned}\nabla(\lambda f + \mu g) &= \lambda \nabla f + \mu \nabla g\\ \nabla \cdot (\lambda \mathbf{F} + \mu \mathbf{G}) &= \lambda \nabla \cdot \mathbf{F} + \mu \nabla \cdot \mathbf{G}\\ \nabla \times (\lambda \mathbf{F} + \mu \mathbf{G}) &= \lambda \nabla \times \mathbf{F} + \mu \nabla \times \mathbf{G}.\end{aligned}$$

Proposition. We have the following Leibnitz properties:

$$\begin{aligned}\nabla(fg) &= (\nabla f)g + f(\nabla g)\\ \nabla \cdot (f\mathbf{F}) &= (\nabla f) \cdot \mathbf{F} + f(\nabla \cdot \mathbf{F})\\ \nabla \times (f\mathbf{F}) &= (\nabla f) \times \mathbf{F} + f(\nabla \times \mathbf{F})\\ \nabla(\mathbf{F} \cdot \mathbf{G}) &= \mathbf{F} \times (\nabla \times \mathbf{G}) + \mathbf{G} \times (\nabla \times \mathbf{F}) + (\mathbf{F} \cdot \nabla)\mathbf{G} + (\mathbf{G} \cdot \nabla)\mathbf{F}\\ \nabla \times (\mathbf{F} \times \mathbf{G}) &= \mathbf{F}(\nabla \cdot \mathbf{G}) - \mathbf{G}(\nabla \cdot \mathbf{F}) + (\mathbf{G} \cdot \nabla)\mathbf{F} - (\mathbf{F} \cdot \nabla)\mathbf{G}\\ \nabla \cdot (\mathbf{F} \times \mathbf{G}) &= (\nabla \times \mathbf{F}) \cdot \mathbf{G} - \mathbf{F} \cdot (\nabla \times \mathbf{G})\end{aligned}$$

which can be proven by brute-forcing with suffix notation and summation convention.

6.2 Second-order derivatives

Proposition.

$$\begin{aligned}\nabla \times (\nabla f) &= 0\\ \nabla \cdot (\nabla \times \mathbf{F}) &= 0\end{aligned}$$

Proof. Expand out using suffix notation, noting that

$$\varepsilon_{ijk} \frac{\partial^2 f}{\partial x_i \partial x_j} = 0.$$

since if, say, $k = 3$, then

$$\varepsilon_{ijk} \frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_1 \partial x_2} - \frac{\partial^2 f}{\partial x_2 \partial x_1} = 0.$$

□

Proposition. If $\mathbf{F}$ is defined in all of $\mathbb{R}^3$, then

$$\nabla \times \mathbf{F} = 0 \Rightarrow \mathbf{F} = \nabla f$$

for some f.

Proposition. If $\mathbf{H}$ is defined over all of $\mathbb{R}^3$ and $\nabla \cdot \mathbf{H} = 0$, then $\mathbf{H} = \nabla \times \mathbf{A}$ for some $\mathbf{A}$.

7 Integral theorems

7.1 Statement and examples

7.1.1 Green's theorem (in the plane)

Theorem (Green's theorem)**.** For smooth functions $P(x, y)$, $Q(x, y)$ and A a bounded region in the (x, y) plane with boundary $\partial A = C$,

$$\int_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \mathrm{d}A = \int_C (P \; \mathrm{d}x + Q \; \mathrm{d}y).$$

Here C is assumed to be piecewise smooth, non-intersecting closed curve, traversed anti-clockwise.

7.1.2 Stokes' theorem

Theorem (Stokes' theorem)**.** For a smooth vector field $\mathbf{F}(\mathbf{r})$,

$$\int_S \nabla \times \mathbf{F} \cdot \mathrm{d}\mathbf{S} = \int_{\partial S} \mathbf{F} \cdot \mathrm{d}\mathbf{r},$$

where S is a smooth, bounded surface and ∂S is a piecewise smooth boundary of S.

The direction of the line integral is as follows: If we walk along C with $\mathbf{n}$ facing up, then the surface is on your left.

7.1.3 Divergence/Gauss theorem

Theorem (Divergence/Gauss theorem)**.** For a smooth vector field $\mathbf{F}(\mathbf{r})$,

$$\int_V \nabla \cdot \mathbf{F} \; \mathrm{d}V = \int_{\partial V} \mathbf{F} \cdot \mathrm{d}\mathbf{S},$$

where V is a bounded volume with boundary ∂V, a piecewise smooth, closed surface, with outward normal $\mathbf{n}$.

7.2 Relating and proving integral theorems

Proposition. Stokes' theorem $\Rightarrow$ Green's theorem

Proof. Stokes' theorem talks about 3D surfaces and Green's theorem is about 2D regions. So given a region A on the (x, y) plane, we pretend that there is a third dimension and apply Stokes' theorem to derive Green's theorem.

Let A be a region in the (x, y) plane with boundary $C = \partial A$, parametrised by arc length, $(x(s), y(s), 0)$. Then the tangent to C is

$$\mathbf{t} = \left(\frac{\mathrm{d}x}{\mathrm{d}s}, \frac{\mathrm{d}y}{\mathrm{d}s}, 0\right).$$

Given any $P(x, y)$ and $Q(x, y)$, we can consider the vector field

$$\mathbf{F} = (P, Q, 0),$$

So

$$\nabla \times \mathbf{F} = \left(0, 0, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right).$$

Then the left hand side of Stokes is

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_C \mathbf{F} \cdot \mathbf{t}\ \mathrm{d}s = \int_C P\ \mathrm{d}x + Q\ \mathrm{d}y,$$

and the right hand side is

$$\int_A (\nabla \times \mathbf{F}) \cdot \hat{\mathbf{k}}\ \mathrm{d}A = \int_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\ \mathrm{d}A. \qquad \square$$

Proposition. Green's theorem $\Rightarrow$ Stokes' theorem.

Proof. Green's theorem describes a 2D region, while Stokes' theorem describes a 3D surface $\mathbf{r}(u, v)$. Hence to use Green's to derive Stokes' we need find some 2D thing to act on. The natural choice is the parameter space, u, v.

Consider a parametrised surface $S = \mathbf{r}(u, v)$ corresponding to the region A in the u, v plane. Write the boundary as $\partial A = (u(t), v(t))$. Then $\partial S = \mathbf{r}(u(t), v(t))$.

We want to prove

$$\int_{\partial S} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_S (\nabla \times \mathbf{F}) \cdot \mathrm{d}\mathbf{S}$$

given

$$\int_{\partial A} F_u\ \mathrm{d}u + F_v\ \mathrm{d}v = \int_A \left(\frac{\partial F_v}{\partial u} - \frac{\partial F_u}{\partial v}\right)\ \mathrm{d}A.$$

Doing some pattern-matching, we want

$$\mathbf{F} \cdot \mathrm{d}\mathbf{r} = F_u\ \mathrm{d}u + F_v\ \mathrm{d}v$$

for some F_u and F_v.

By the chain rule, we know that

$$\mathrm{d}\mathbf{r} = \frac{\partial \mathbf{r}}{\partial u}\mathrm{d}u + \frac{\partial \mathbf{r}}{\partial v}\mathrm{d}v.$$

So we choose

$$F_u = \mathbf{F} \cdot \frac{\partial \mathbf{r}}{\partial u}, \quad F_v = \mathbf{F} \cdot \frac{\partial \mathbf{r}}{\partial v}.$$

This choice matches the left hand sides of the two equations.

To match the right, recall that

$$(\nabla \times \mathbf{F}) \cdot \mathrm{d}\mathbf{S} = (\nabla \times \mathbf{F}) \cdot \left(\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right)\ \mathrm{d}u\ \mathrm{d}v.$$

Therefore, for the right hand sides to match, we want

$$\frac{\partial F_v}{\partial u} - \frac{\partial F_u}{\partial v} = (\nabla \times \mathbf{F}) \cdot \left(\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right). \qquad (*)$$

Fortunately, this is true. Unfortunately, the proof involves complicated suffix notation and summation convention:

$$\frac{\partial F_v}{\partial u} = \frac{\partial}{\partial u}\left(\mathbf{F} \cdot \frac{\partial \mathbf{r}}{\partial v}\right) = \frac{\partial}{\partial u}\left(F_i \frac{\partial x_i}{\partial v}\right) = \left(\frac{\partial F_i}{\partial x_j}\frac{\partial x_j}{\partial u}\right)\frac{\partial x_i}{\partial v} + F_i \frac{\partial x_i}{\partial u \partial v}.$$

Similarly,

$$\frac{\partial F_u}{\partial u} = \frac{\partial}{\partial u}\left(\mathbf{F}\cdot\frac{\partial \mathbf{r}}{\partial u}\right) = \frac{\partial}{\partial u}\left(F_j\frac{\partial x_j}{\partial u}\right) = \left(\frac{\partial F_j}{\partial x_i}\frac{\partial x_i}{\partial v}\right)\frac{\partial x_j}{\partial u} + F_i\frac{\partial x_i}{\partial u \partial v}.$$

So

$$\frac{\partial F_v}{\partial u} - \frac{\partial F_u}{\partial v} = \frac{\partial x_j}{\partial u}\frac{\partial x_i}{\partial v}\left(\frac{\partial F_i}{\partial x_j} - \frac{\partial F_j}{\partial x_i}\right).$$

This is the left hand side of $(*)$.

The right hand side of $(*)$ is

$$\begin{aligned}(\nabla\times\mathbf{F})\cdot\left(\frac{\partial \mathbf{r}}{\partial u}\times\frac{\partial \mathbf{r}}{\partial v}\right) &= \varepsilon_{ijk}\frac{\partial F_j}{\partial x_i}\varepsilon_{kpq}\frac{\partial x_p}{\partial u}\frac{\partial x_q}{\partial v}\\ &= (\delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp})\frac{\partial F_j}{\partial x_i}\frac{\partial x_p}{\partial u}\frac{\partial x_q}{\partial v}\\ &= \left(\frac{\partial F_j}{\partial x_i} - \frac{\partial F_i}{\partial x_j}\right)\frac{\partial x_i}{\partial u}\frac{\partial x_j}{\partial v}.\end{aligned}$$

So they match. Therefore, given our choice of F_u and F_v, Green's theorem translates to Stokes' theorem. □

Proposition. Greens theorem $\Leftrightarrow$ 2D divergence theorem.

Proof. The 2D divergence theorem states that

$$\int_A (\nabla\cdot\mathbf{G})\;\mathrm{d}A = \int_{\partial A}\mathbf{G}\cdot\mathbf{n}\;\mathrm{d}s.$$

with an outward normal $\mathbf{n}$.

Write $\mathbf{G}$ as $(Q, -P)$. Then

$$\nabla\cdot\mathbf{G} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}.$$

Around the curve $\mathbf{r}(s) = (x(s), y(s))$, $\mathbf{t}(s) = (x'(s), y'(s))$. Then the normal, being tangent to $\mathbf{t}$, is $\mathbf{n}(s) = (y'(s), -x'(s))$ (check that it points outwards!). So

$$\mathbf{G}\cdot\mathbf{n} = P\frac{\mathrm{d}x}{\mathrm{d}s} + Q\frac{\mathrm{d}y}{\mathrm{d}s}.$$

Then we can expand out the integrals to obtain

$$\int_C \mathbf{G}\cdot\mathbf{n}\;\mathrm{d}s = \int_C P\;\mathrm{d}x + Q\;\mathrm{d}y,$$

and

$$\int_A (\nabla\cdot\mathbf{G})\;\mathrm{d}A = \int_A\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\;\mathrm{d}A.$$

Now 2D version of Gauss' theorem says the two LHS are the equal, and Green's theorem says the two RHS are equal. So the result follows. □

Proposition. 2D divergence theorem.

$$\int_A (\nabla\cdot\mathbf{G})\;\mathrm{d}A = \int_{C=\partial A}\mathbf{G}\cdot\mathbf{n}\;\mathrm{d}s.$$

Proof. For the sake of simplicity, we assume that $\mathbf{G}$ only has a vertical component, noting that the same proof works for purely horizontal $\mathbf{G}$, and an arbitrary $\mathbf{G}$ is just a linear combination of the two.

Furthermore, we assume that A is a simple, convex shape. A more complicated shape can be cut into smaller simple regions, and we can apply the simple case to each of the small regions.

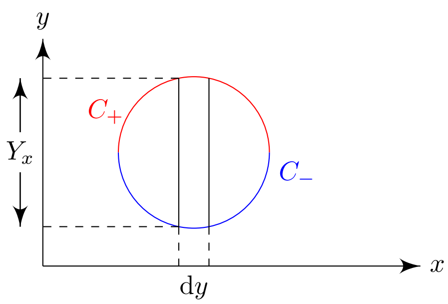
Suppose $\mathbf{G} = G(x, y)\hat{\mathbf{j}}$. Then

$$\nabla \cdot \mathbf{G} = \frac{\partial G}{\partial y}.$$

Then

$$\int_A \nabla \cdot \mathbf{G} \, \mathrm{d}A = \int_X \left(\int_{Y_x} \frac{\partial G}{\partial y} \, \mathrm{d}y\right) \, \mathrm{d}x.$$

Now we divide A into an upper and lower part, with boundaries $C_+ = y_+(x)$ and $C_- = y_-(x)$ respectively. Since I cannot draw, A will be pictured as a circle, but the proof is valid for any simple convex shape.



We see that the boundary of Y_x at any specific x is given by $y_-(x)$ and $y_+(x)$. Hence by the Fundamental theorem of Calculus,

$$\int_{Y_x} \frac{\partial G}{\partial y} \, \mathrm{d}y = \int_{y_-(x)}^{y_+(x)} \frac{\partial G}{\partial y} \, \mathrm{d}y = G(x, y_+(x)) - G(x, y_-(x)).$$

To compute the full area integral, we want to integrate over all x. However, the divergence theorem talks in terms of $\mathrm{d}s$, not $\mathrm{d}x$. So we need to find some way to relate $\mathrm{d}s$ and $\mathrm{d}x$. If we move a distance δs, the change in x is $\delta s \cos\theta$, where θ is the angle between the tangent and the horizontal. But θ is also the angle between the normal and the vertical. So $\cos\theta = \mathbf{n} \cdot \hat{\mathbf{j}}$. Therefore $\mathrm{d}x = \hat{\mathbf{j}} \cdot \mathbf{n} \, \mathrm{d}s$.

In particular, $G \, \mathrm{d}x = G \, \hat{\mathbf{j}} \cdot \mathbf{n} \, \mathrm{d}s = \mathbf{G} \cdot \mathbf{n} \, \mathrm{d}s$, since $\mathbf{G} = G \, \hat{\mathbf{j}}$.

However, at C_-, $\mathbf{n}$ points downwards, so $\mathbf{n} \cdot \hat{\mathbf{j}}$ happens to be negative. So, actually, at C_-, $\mathrm{d}x = -\mathbf{G} \cdot \mathbf{n} \, \mathrm{d}s$.

Therefore, our full integral is

$$\begin{aligned}
\int_A \nabla \cdot \mathbf{G}\ \mathrm{d}A &= \int_X \left(\int_{y_x} \frac{\partial G}{\partial y}\ \mathrm{d}Y\right)\ \mathrm{d}x\\
&= \int_X G(x, y_+(x)) - G(x, y_-(x))\ \mathrm{d}x\\
&= \int_{C_+} \mathbf{G}\cdot\mathbf{n}\ \mathrm{d}s + \int_{C_-} \mathbf{G}\cdot\mathbf{n}\ \mathrm{d}s\\
&= \int_C \mathbf{G}\cdot\mathbf{n}\ \mathrm{d}s. \qquad \square
\end{aligned}$$

8 Some applications of integral theorems

8.1 Integral expressions for div and curl

Proposition.

$$(\nabla \cdot \mathbf{F})(\mathbf{r}_0) = \lim_{\mathrm{diam}(V)\to 1} \frac{1}{\mathrm{vol}(V)} \int_{\partial V} \mathbf{F} \cdot \mathrm{d}\mathbf{S},$$

where the limit is taken over volumes containing the point $\mathbf{r}_0$.

Proposition.

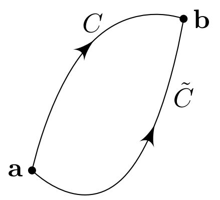
$$\mathbf{n} \cdot (\nabla \times \mathbf{F})(\mathbf{r}_0) = \lim_{\mathrm{diam}(A)\to 0} \frac{1}{\mathrm{area}(A)} \int_{\partial A} \mathbf{F} \cdot \mathrm{d}\mathbf{r},$$

where the limit is taken over all surfaces A containing $\mathbf{r}_0$ with normal $\mathbf{n}$.

8.2 Conservative fields and scalar products

Proposition. If (iii) $\nabla \times \mathbf{F} = 0$, then (ii) $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$ is independent of C.

Proof. Given $\mathbf{F}(\mathbf{r})$ satisfying $\nabla \times \mathbf{F} = 0$, let C and $\tilde{C}$ be any two curves from $\mathbf{a}$ to $\mathbf{b}$.



If S is any surface with boundary $\partial S = C - \tilde{C}$, By Stokes' theorem,

$$\int_S \nabla \times \mathbf{F} \cdot \mathrm{d}\mathbf{S} = \int_{\partial S} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} - \int_{\tilde{C}} \mathbf{F} \cdot \mathrm{d}\mathbf{r}.$$

But $\nabla \times \mathbf{F} = 0$. So

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} - \int_{\tilde{C}} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = 0,$$

or

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_{\tilde{C}} \mathbf{F} \cdot \mathrm{d}\mathbf{r}.$$

□

Proposition. If (ii) $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$ is independent of C for fixed end points and orientation, then (i) $\mathbf{F} = \nabla f$ for some scalar field f.

Proof. We fix $\mathbf{a}$ and define $f(\mathbf{r}) = \int_C \mathbf{F}(\mathbf{r}') \cdot \mathrm{d}\mathbf{r}'$ for any curve from $\mathbf{a}$ to $\mathbf{r}$. Assuming (ii), f is well-defined. For small changes $\mathbf{r}$ to $\mathbf{r} + \delta\mathbf{r}$, there is a small extension of C by δC. Then

$$\begin{aligned} f(\mathbf{r} + \delta\mathbf{r}) &= \int_{C+\delta C} \mathbf{F}(\mathbf{r}') \cdot \mathrm{d}\mathbf{r}' \\ &= \int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}' + \int_{\delta C} \mathbf{F} \cdot \mathrm{d}\mathbf{r}' \\ &= f(\mathbf{r}) + \mathbf{F}(\mathbf{r}) \cdot \delta\mathbf{r} + o(\delta\mathbf{r}). \end{aligned}$$

So
$$\delta f = f(\mathbf{r} + \delta\mathbf{r}) - f(\mathbf{r}) = \mathbf{F}(\mathbf{r}) \cdot \delta\mathbf{r} + o(\delta\mathbf{r}).$$
But the definition of grad is exactly
$$\delta f = \nabla f \cdot \delta\mathbf{r} + o(\delta\mathbf{r}).$$
So we have $\mathbf{F} = \nabla f$. $\square$

8.3 Conservation laws

9 Orthogonal curvilinear coordinates

9.1 Line, area and volume elements

9.2 Grad, Div and Curl

Proposition.

$$\nabla f = \frac{1}{h_u}\frac{\partial f}{\partial u}\mathbf{e}_u + \frac{1}{h_v}\frac{\partial f}{\partial v}\mathbf{e}_v + \frac{1}{h_w}\frac{\partial f}{\partial w}\mathbf{e}_w.$$

Proposition.

$$\nabla = \frac{1}{h_u}\mathbf{e}_u\frac{\partial}{\partial u} + \frac{1}{h_v}\mathbf{e}_v\frac{\partial}{\partial v} + \frac{1}{h_w}\mathbf{e}_w\frac{\partial}{\partial w}.$$

Proposition.

$$\begin{aligned}\nabla\times\mathbf{F} &= \frac{1}{h_v h_w}\left[\frac{\partial}{\partial v}(h_w F_w) - \frac{\partial}{\partial w}(h_v F_v)\right]\mathbf{e}_u + \text{ two similar terms}\\ &= \frac{1}{h_u h_v h_w}\begin{vmatrix} h_u\mathbf{e}_u & h_v\mathbf{e}_v & h_w\mathbf{e}_w\\ \frac{\partial}{\partial u} & \frac{\partial}{\partial v} & \frac{\partial}{\partial w}\\ h_u F_u & h_v F_v & h_w F_w\end{vmatrix}\end{aligned}$$

and

$$\nabla\cdot\mathbf{F} = \frac{1}{h_u h_v h_w}\left[\frac{\partial}{\partial u}(h_v h_w F_u) + \text{ two similar terms}\right].$$

Proof. (non-examinable)

(i) Apply $\nabla\cdot$ or $\nabla\times$ and differentiate the basis vectors explicitly.

(ii) First, apply $\nabla\cdot$ or $\nabla\times$, but calculate the result by writing $\mathbf{F}$ in terms of $\nabla u, \nabla v$ and ∇w in a suitable way. Then use $\nabla\times\nabla f = 0$ and $\nabla\cdot(\nabla\times f) = 0$.

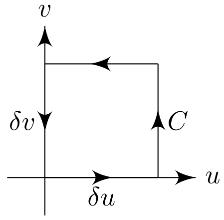
(iii) Use the integral expressions for div and curl.

Recall that

$$\mathbf{n}\cdot\nabla\times\mathbf{F} = \lim_{A\to 0}\frac{1}{A}\int_{\partial A}\mathbf{F}\cdot\mathrm{d}\mathbf{r}.$$

So to calculate the curl, we first find the $\mathbf{e}_w$ component.

Consider an area with W fixed and change u by δu and v by δv. Then this has an area of $h_u h_v \delta u \delta v$ with normal $\mathbf{e}_w$. Let C be its boundary.



We then integrate around the curve C. We split the curve C up into 4 parts (corresponding to the four sides), and take linear approximations by assuming F and h are constant when moving through each horizontal/vertical segment.

$$
\begin{aligned}
\int_C \mathbf{F}\cdot \mathrm{d}\mathbf{r} &\approx F_u(u,v)h_u(u,v)\ \delta u + F_v(u+\delta u, v)h_v(u+\delta u, v)\ \delta u\\
&\quad - F_u(u, v+\delta v)h_u(u, v+\delta v)\ \delta u - F_v(u,v)h_v(u,v)\ \delta v\\
&\approx \left[\frac{\partial}{\partial u} h_v F_v - \frac{\partial}{\partial v}(h_u F_u)\right]\delta u\delta v.
\end{aligned}
$$

Divide by the area and take the limit as area $\to 0$, we obtain

$$
\lim_{A\to 0}\frac{1}{A}\int_C \mathbf{F}\cdot \mathrm{d}\mathbf{r} = \frac{1}{h_u h_v}\left[\frac{\partial}{\partial u} h_v F_v - \frac{\partial}{\partial v}(h_u F_u)\right].
$$

So, by the integral definition of divergence,

$$
\mathbf{e}_w \cdot \nabla \times \mathbf{F} = \frac{1}{h_u h_v}\left[\frac{\partial}{\partial u}(h_v F_v) - \frac{\partial}{\partial v}(h_u F_u)\right],
$$

and similarly for other components.

We can find the divergence similarly. □

10 Gauss' Law and Poisson's equation

10.1 Laws of gravitation

Law (Gauss' law for gravitation)**.** Given any volume V bounded by closed surface S,

$$\int_S \mathbf{g} \cdot \mathrm{d}\mathbf{S} = -4\pi GM,$$

where G is Newton's gravitational constant, and M is the total mass contained in V.

Law (Gauss' Law for gravitation in differential form)**.**

$$\nabla \cdot \mathbf{g} = -4\pi G\rho.$$

10.2 Laws of electrostatics

Law (Gauss' law for electrostatic forces)**.**

$$\int_S \mathbf{E} \cdot \mathrm{d}\mathbf{S} = \frac{Q}{\varepsilon_0},$$

where ε_0 is the *permittivity of free space*, or *electric constant.*

Law (Gauss' law for electrostatic forces in differential form)**.**

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}.$$

10.3 Poisson's Equation and Laplace's equation

11 Laplace's and Poisson's equations

11.1 Uniqueness theorems

Theorem. Consider $\nabla^2\varphi = -\rho$ for some $\rho(\mathbf{r})$ on a bounded volume V with $S = \partial V$ being a closed surface, with an outward normal $\mathbf{n}$.

Suppose φ satisfies either

(i) Dirichlet condition, $\varphi(\mathbf{r}) = f(\mathbf{r})$ on S

(ii) Neumann condition $\frac{\partial \varphi(\mathbf{r})}{\partial \mathbf{n}} = n \cdot \nabla\varphi = g(\mathbf{r})$ on S.

where f, g are given. Then

(i) $\varphi(\mathbf{r})$ is unique

(ii) $\varphi(\mathbf{r})$ is unique up to a constant.

Proof. Let $\varphi_1(\mathbf{r})$ and $\varphi_2(\mathbf{r})$ satisfy Poisson's equation, each obeying the boundary conditions (N) or (D). Then $\Psi(\mathbf{r}) = \varphi_2(\mathbf{r}) - \varphi_1(\mathbf{r})$ satisfies $\nabla^2\Psi = 0$ on V by linearity, and

(i) $\Psi = 0$ on S; or

(ii) $\frac{\partial \Psi}{\partial \mathbf{n}} = 0$ on S.

Combining these two together, we know that $\Psi \frac{\partial \Psi}{\partial \mathbf{n}} = 0$ on the surface. So using the divergence theorem,

$$\int_V \nabla \cdot (\Psi\nabla\Psi)\,\mathrm{d}V = \int_S (\Psi\nabla\Psi)\cdot \mathrm{d}\mathbf{S} = 0.$$

But

$$\nabla\cdot(\Psi\nabla\Psi) = (\nabla\Psi)\cdot(\nabla\Psi) + \Psi\underbrace{\nabla^2\Psi}_{=0} = |(\nabla\Psi)|^2.$$

So

$$\int_V |\nabla\Psi|^2\,\mathrm{d}V = 0.$$

Since $|\nabla\Psi|^2 \geq 0$, the integral can only vanish if $|\nabla\Psi| = 0$. So $\nabla\Psi = 0$. So $\Psi = c$, a constant on V. So

(i) $\Psi = 0$ on $S \Rightarrow c = 0$. So $\varphi_1 = \varphi_2$ on V.

(ii) $\varphi_2(\mathbf{r}) = \varphi_1(\mathbf{r}) + C$, as claimed. □

Proposition (Green's first identity)**.**

$$\int_S (u\nabla v)\cdot \mathrm{d}S = \int_V (\nabla u)\cdot(\nabla v)\,\mathrm{d}V + \int_V u\nabla^2 v\,\mathrm{d}V,$$

Proposition (Green's second identity)**.**

$$\int_S (u\nabla v - v\nabla u)\cdot \mathrm{d}\mathbf{S} = \int_V (u\nabla^2 v - v\nabla^2 u)\,\mathrm{d}V.$$

11.2 Laplace's equation and harmonic functions

11.2.1 The mean value property

Proposition (Mean value property)**.** Suppose $\varphi(\mathbf{r})$ is harmonic on region V containing a solid sphere defined by $|\mathbf{r}-\mathbf{a}| \leq R$, with boundary $S_R = |\mathbf{r}-\mathbf{a}| = R$, for some R. Define

$$\bar{\varphi}(R) = \frac{1}{4\pi R^2}\int_{S_R} \varphi(\mathbf{r})\ \mathrm{d}S.$$

Then $\varphi(\mathbf{a}) = \bar{\varphi}(R)$.

Proof. Note that $\bar{\varphi}(R) \to \varphi(\mathbf{a})$ as $R \to 0$. We take spherical coordinates (u, θ, χ) centered on $\mathbf{r} = \mathbf{a}$. The scalar element (when $u = R$) on S_R is

$$\mathrm{d}S = R^2 \sin\theta\ \mathrm{d}\theta\ \mathrm{d}\chi.$$

So $\frac{\mathrm{d}S}{R^2}$ is independent of R. Write

$$\bar{\varphi}(R) = \frac{1}{4\pi}\int \varphi\ \frac{\mathrm{d}S}{R^2}.$$

Differentiate this with respect to R, noting that $\mathrm{d}S/R^2$ is independent of R. Then we obtain

$$\frac{\mathrm{d}}{\mathrm{d}R}\bar{\varphi}(R) = \frac{1}{4\pi R^2}\int \left.\frac{\partial\varphi}{\partial u}\right|_{u=R} \mathrm{d}S$$

But

$$\frac{\partial\varphi}{\partial u} = \mathbf{e}_u \cdot \nabla\varphi = \mathbf{n}\cdot\nabla\varphi = \frac{\partial\varphi}{\partial\mathbf{n}}$$

on S_R. So

$$\frac{\mathrm{d}}{\mathrm{d}R}\bar{\varphi}(R) = \frac{1}{4\pi R^2}\int_{S_R}\nabla\varphi\cdot\mathrm{d}\mathbf{S} = \frac{1}{4\pi R^2}\int_{V_R}\nabla^2\varphi\ \mathrm{d}V = 0$$

by divergence theorem. So $\bar{\varphi}(R)$ does not depend on R, and the result follows. □

11.2.2 The maximum (or minimum) principle

Proposition (Maximum principle)**.** If a function φ is harmonic on a region V, then φ cannot have a maximum at an interior point of $\mathbf{a}$ of V.

Proof. Suppose that φ had a local maximum at $\mathbf{a}$ in the interior. Then there is an ε such that for any $\mathbf{r}$ such that $0 < |\mathbf{r}-\mathbf{a}| < \varepsilon$, we have $\varphi(\mathbf{r}) < \varphi(\mathbf{a})$.

Note that if there is an ε that works, then any smaller ε will work. Pick an ε sufficiently small such that the region $|\mathbf{r}-\mathbf{a}| < \varepsilon$ lies within V (possible since $\mathbf{a}$ lies in the interior of V).

Then for any $\mathbf{r}$ such that $|\mathbf{r}-\mathbf{a}| = \varepsilon$, we have $\varphi(\mathbf{r}) < \varphi(\mathbf{a})$.

$$\bar{\varphi}(\varepsilon) = \frac{1}{4\pi R^2}\int_{S_R}\varphi(\mathbf{r})\ \mathrm{d}S < \varphi(\mathbf{a}),$$

which contradicts the mean value property. □

11.3 Integral solutions of Poisson's equations

11.3.1 Statement and informal derivation

Proposition. The solution to Poisson's equation $\nabla^2\varphi = -\rho$, with boundary conditions $|\varphi(\mathbf{r})| = O(1/|\mathbf{r}|)$ and $|\nabla\varphi(\mathbf{r})| = O(1/|\mathbf{r}|^2)$, is

$$\varphi(\mathbf{r}) = \frac{1}{4\pi}\int_{V'} \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,\mathrm{d}V'$$

For $\rho(\mathbf{r}')$ non-zero everywhere, but suitably well-behaved as $|\mathbf{r}'| \to \infty$, we can also take $V' = \mathbb{R}^3$.

11.3.2 Point sources and δ-functions*

12 Maxwell's equations

12.1 Laws of electromagnetism

Law (Lorentz force law)**.** A point charge q experiences a force of

$$\mathbf{F} = q(\mathbf{E} + \dot{\mathbf{r}} \times \mathbf{B}).$$

Law (Maxwell's equations)**.**

$$\begin{aligned}
\nabla \cdot \mathbf{E} &= \frac{\rho}{\varepsilon_0} \\
\nabla \cdot \mathbf{B} &= 0 \\
\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} &= 0 \\
\nabla \times \mathbf{B} - \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} &= \mu_0 \mathbf{j},
\end{aligned}$$

where ε_0 is the electric constant (permittivity of free space) and μ_0 is the magnetic constant (permeability of free space), which are constants determined experimentally.

12.2 Static charges and steady currents

12.3 Electromagnetic waves

13 Tensors and tensor fields

13.1 Definition

13.2 Tensor algebra

13.3 Symmetric and antisymmetric tensors

13.4 Tensors, multi-linear maps and the quotient rule

Proposition (Quotient rule)**.** Suppose that $T_{i\cdots jp\cdots q}$ is an array defined in each coordinate system, and that $v_{i\cdots j} = T_{i\cdots jp\cdots q}u_{p\cdots q}$ is also a tensor for any tensor $u_{p\cdots q}$. Then $T_{i\cdots jp\cdots q}$ is also a tensor.

Proof. We can check the tensor transformation rule directly. However, we can reuse the result above to save some writing.

Consider the special form $u_{p\cdots q} = c_p \cdots d_q$ for any vectors $\mathbf{c}, \cdots \mathbf{d}$. By assumption,

$$v_{i\cdots j} = T_{i\cdots jp\cdots q}c_p \cdots d_q$$

is a tensor. Then

$$v_{i\cdots j}a_i \cdots b_j = T_{i\cdots jp\cdots q}a_i \cdots b_j c_p \cdots d_q$$

is a scalar for any vectors $\mathbf{a}, \cdots, \mathbf{b}, \mathbf{c}, \cdots, \mathbf{d}$. Since $T_{i\cdots jp\cdots q}a_i \cdots b_j c_p \cdots d_q$ is a scalar and hence gives the same result in every coordinate system, $T_{i\cdots jp\cdots q}$ is a multi-linear map. So $T_{i\cdots jp\cdots q}$ is a tensor. □

13.5 Tensor calculus

Proposition.

$$\underbrace{\frac{\partial}{\partial x_p} \cdots \frac{\partial}{\partial x_q}}_{m} T_{\underbrace{ij\cdots k}_{n}}, \tag{$*$}$$

is a tensor of rank $n + m$.

Proof. To show this, it suffices to show that $\frac{\partial}{\partial x_p}$ satisfies the tensor transformation rules for rank 1 tensors (i.e. it is something like a rank 1 tensor). Then by the exact same argument we used to show that tensor products preserve tensorness, we can show that the $(*)$ is a tensor. (we cannot use the result of tensor products directly, since this is not exactly a product. But the exact same proof works!)

Since $x'_i = R_{iq}x_q$, we have

$$\frac{\partial x'_i}{\partial x_p} = R_{ip}.$$

(noting that $\frac{\partial x_p}{\partial x_q} = \delta_{pq}$). Similarly,

$$\frac{\partial x_q}{\partial x'_i} = R_{iq}.$$

Note that R_{ip}, R_{iq} are constant matrices.

Hence by the chain rule,

$$\frac{\partial}{\partial x_i'} = \left(\frac{\partial x_q}{\partial x_i'}\right)\frac{\partial}{\partial x_q} = R_{iq}\frac{\partial}{\partial x_q}.$$

So $\frac{\partial}{\partial x_p}$ obeys the vector transformation rule. So done. $\square$

Theorem (Divergence theorem for tensors)**.**

$$\int_S T_{ij\cdots k\ell} n_\ell \;\mathrm{d}S = \int_V \frac{\partial}{\partial x_\ell}(T_{ij\cdots k\ell})\;\mathrm{d}V,$$

with $\mathbf{n}$ being an outward pointing normal.

Proof. Apply the usual divergence theorem to the vector field $\mathbf{v}$ defined by $v_\ell = a_i b_j \cdots c_k T_{ij\cdots k\ell}$, where $\mathbf{a}, \mathbf{b}, \cdots, \mathbf{c}$ are fixed constant vectors.

Then

$$\nabla\cdot\mathbf{v} = \frac{\partial v_\ell}{\partial x_\ell} = a_i b_j \cdots c_k \frac{\partial}{\partial x_\ell} T_{ij\cdots k\ell},$$

and

$$\mathbf{n}\cdot\mathbf{v} = n_\ell v_\ell = a_i b_j \cdots c_k T_{ij\cdots k\ell} n_\ell.$$

Since $\mathbf{a}, \mathbf{b}, \cdots, \mathbf{c}$ are arbitrary, therefore they can be eliminated, and the tensor divergence theorem follows. $\square$

14 Tensors of rank 2

14.1 Decomposition of a second-rank tensor

14.2 The inertia tensor

14.3 Diagonalization of a symmetric second rank tensor

15 Invariant and isotropic tensors

15.1 Definitions and classification results

Theorem.

(i) There are no isotropic tensors of rank 1, except the zero tensor.

(ii) The most general rank 2 isotropic tensor is $T_{ij} = \alpha\delta_{ij}$ for some scalar α.

(iii) The most general rank 3 isotropic tensor is $T_{ijk} = \beta\varepsilon_{ijk}$ for some scalar β.

(iv) All isotropic tensors of higher rank are obtained by combining δ_{ij} and ε_{ijk} using tensor products, contractions, and linear combinations.

Proof. We analyze conditions for invariance under specific rotations through π or $\pi/2$ about coordinate axes.

(i) Suppose T_i is rank-1 isotropic. Consider a rotation about x_3 through π:

$$(R_{ij}) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We want $T_1 = R_{ip}T_p = R_{11}T_1 = -T_1$. So $T_1 = 0$. Similarly, $T_2 = 0$. By consider a rotation about, say x_1, we have $T_3 = 0$.

(ii) Suppose T_{ij} is rank-2 isotropic. Consider

$$(R_{ij}) = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

which is a rotation through $\pi/2$ about the x_3 axis. Then

$$T_{13} = R_{1p}R_{3q}T_{pq} = R_{12}R_{33}T_{23} = T_{23}$$

and

$$T_{23} = R_{2p}R_{3q}T_{pq} = R_{21}R_{33}T_{13} = -T_{13}$$

So $T_{13} = T_{23} = 0$. Similarly, we have $T_{31} = T_{32} = 0$.

We also have

$$T_{11} = R_{1p}R_{1q}T_{pq} = R_{12}R_{12}T_{22} = T_{22}.$$

So $T_{11} = T_{22}$.

By picking a rotation about a different axis, we have $T_{21} = T_{12}$ and $T_{22} = T_{33}$.

Hence $T_{ij} = \alpha\delta_{ij}$.

(iii) Suppose that T_{ijk} is rank-3 isotropic. Using the rotation by π about the x_3 axis, we have

$$T_{133} = R_{1p}R_{3q}R_{3r}T_{pqr} = -T_{133}.$$

So $T_{133} = 0$. We also have

$$T_{111} = R_{1p}R_{1q}R_{1r}T_{pqr} = -T_{111}.$$

So $T_{111} = 0$. We have similar results for π rotations about other axes and other choices of indices.

Then we can show that $T_{ijk} = 0$ unless all i, j, k are distinct.

Now consider

$$(R_{ij}) = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

a rotation about x_3 through $\pi/2$. Then

$$T_{123} = R_{1p}R_{2q}R_{3r}T_{pqr} = R_{12}R_{21}R_{33}T_{213} = -T_{213}.$$

So $T_{123} = -T_{213}$. Along with similar results for other indices and axes of rotation, we find that T_{ijk} is totally antisymmetric, and $T_{ijk} = \beta\varepsilon_{ijk}$ for some β. □

15.2 Application to invariant integrals

Theorem. Let

$$T_{ij\cdots k} = \int_V f(\mathbf{x})x_i x_j \cdots x_k \, \mathrm{d}V.$$

where $f(\mathbf{x})$ is a scalar function and V is some volume.

Given a rotation R_{ij}, consider an *active* transformation: $\mathbf{x} = x_i\mathbf{e}_i$ is mapped to $\mathbf{x}' = x_i'\mathbf{e}_i$ with $x_i' = R_{ij}x_i$, i.e. we map the components but not the basis, and V is mapped to V'.

Suppose that under this active transformation,

(i) $f(\mathbf{x}) = f(\mathbf{x}')$,

(ii) $V' = V$ (e.g. if V is all of space or a sphere).

Then $T_{ij\cdots k}$ is invariant under the rotation.

Proof. First note that the Jacobian of the transformation R is 1, since it is simply the determinant of R ($x_i' = R_{ip}x_p \Rightarrow \frac{\partial x_i'}{\partial x_p} = R_{ip}$), which is by definition 1. So $\mathrm{d}V = \mathrm{d}V'$.

Then we have

$$\begin{aligned}
R_{ip}R_{jq}\cdots R_{kr}T_{pq\cdots r} &= \int_V f(\mathbf{x})x_i'x_j'\cdots x_k' \, \mathrm{d}V \\
&= \int_V f(\mathbf{x}')x_i'x_j'\cdots x_k' \, \mathrm{d}V && \text{using (i)} \\
&= \int_{V'} f(\mathbf{x}')x_i'x_j'\cdots x_k' \, \mathrm{d}V' && \text{using (ii)} \\
&= \int_V f(\mathbf{x})x_i x_j\cdots x_k \, \mathrm{d}V && \text{since } x_i \text{ and } x_i' \text{ are dummy} \\
&= T_{ij\cdots k}.
\end{aligned}$$

□