

# Part IA — Vector Calculus

## Theorems

Based on lectures by B. Allanach

Notes taken by Dexter Chua

Lent 2015

These notes are not endorsed by the lecturers, and I have modified them (often significantly) after lectures. They are nowhere near accurate representations of what was actually lectured, and in particular, all errors are almost surely mine.

### Curves in $\mathbb{R}^3$

Parameterised curves and arc length, tangents and normals to curves in  $\mathbb{R}^3$ , the radius of curvature. [1]

### Integration in $\mathbb{R}^2$ and $\mathbb{R}^3$

Line integrals. Surface and volume integrals: definitions, examples using Cartesian, cylindrical and spherical coordinates; change of variables. [4]

### Vector operators

Directional derivatives. The gradient of a real-valued function: definition; interpretation as normal to level surfaces; examples including the use of cylindrical, spherical \*and general orthogonal curvilinear\* coordinates.

Divergence, curl and  $\nabla^2$  in Cartesian coordinates, examples; formulae for these operators (statement only) in cylindrical, spherical \*and general orthogonal curvilinear\* coordinates. Solenoidal fields, irrotational fields and conservative fields; scalar potentials. Vector derivative identities. [5]

### Integration theorems

Divergence theorem, Green's theorem, Stokes's theorem, Green's second theorem: statements; informal proofs; examples; application to fluid dynamics, and to electromagnetism including statement of Maxwell's equations. [5]

### Laplace's equation

Laplace's equation in  $\mathbb{R}^2$  and  $\mathbb{R}^3$ : uniqueness theorem and maximum principle. Solution of Poisson's equation by Gauss's method (for spherical and cylindrical symmetry) and as an integral. [4]

### Cartesian tensors in $\mathbb{R}^3$

Tensor transformation laws, addition, multiplication, contraction, with emphasis on tensors of second rank. Isotropic second and third rank tensors. Symmetric and antisymmetric tensors. Revision of principal axes and diagonalization. Quotient theorem. Examples including inertia and conductivity. [5]

# Contents

<b>0</b>	<b>Introduction</b>	<b>4</b>
<b>1</b>	<b>Derivatives and coordinates</b>	<b>5</b>
1.1	Derivative of functions . . . . .	5
1.2	Inverse functions . . . . .	5
1.3	Coordinate systems . . . . .	5
<b>2</b>	<b>Curves and Line</b>	<b>6</b>
2.1	Parametrised curves, lengths and arc length . . . . .	6
2.2	Line integrals of vector fields . . . . .	6
2.3	Gradients and Differentials . . . . .	6
2.4	Work and potential energy . . . . .	6
<b>3</b>	<b>Integration in <math>\mathbb{R}^2</math> and <math>\mathbb{R}^3</math></b>	<b>7</b>
3.1	Integrals over subsets of $\mathbb{R}^2$ . . . . .	7
3.2	Change of variables for an integral in $\mathbb{R}^2$ . . . . .	7
3.3	Generalization to $\mathbb{R}^3$ . . . . .	7
3.4	Further generalizations . . . . .	8
<b>4</b>	<b>Surfaces and surface integrals</b>	<b>9</b>
4.1	Surfaces and Normal . . . . .	9
4.2	Parametrized surfaces and area . . . . .	9
4.3	Surface integral of vector fields . . . . .	9
4.4	Change of variables in $\mathbb{R}^2$ and $\mathbb{R}^3$ revisited . . . . .	9
<b>5</b>	<b>Geometry of curves and surfaces</b>	<b>10</b>
<b>6</b>	<b>Div, Grad, Curl and <math>\nabla</math></b>	<b>11</b>
6.1	Div, Grad, Curl and $\nabla$ . . . . .	11
6.2	Second-order derivatives . . . . .	11
<b>7</b>	<b>Integral theorems</b>	<b>12</b>
7.1	Statement and examples . . . . .	12
7.1.1	Green's theorem (in the plane) . . . . .	12
7.1.2	Stokes' theorem . . . . .	12
7.1.3	Divergence/Gauss theorem . . . . .	12
7.2	Relating and proving integral theorems . . . . .	12
<b>8</b>	<b>Some applications of integral theorems</b>	<b>13</b>
8.1	Integral expressions for div and curl . . . . .	13
8.2	Conservative fields and scalar products . . . . .	13
8.3	Conservation laws . . . . .	13
<b>9</b>	<b>Orthogonal curvilinear coordinates</b>	<b>14</b>
9.1	Line, area and volume elements . . . . .	14
9.2	Grad, Div and Curl . . . . .	14

---

<b>10 Gauss' Law and Poisson's equation</b>	<b>15</b>
10.1 Laws of gravitation . . . . .	15
10.2 Laws of electrostatics . . . . .	15
10.3 Poisson's Equation and Laplace's equation . . . . .	15
<b>11 Laplace's and Poisson's equations</b>	<b>16</b>
11.1 Uniqueness theorems . . . . .	16
11.2 Laplace's equation and harmonic functions . . . . .	16
11.2.1 The mean value property . . . . .	16
11.2.2 The maximum (or minimum) principle . . . . .	16
11.3 Integral solutions of Poisson's equations . . . . .	16
11.3.1 Statement and informal derivation . . . . .	16
11.3.2 Point sources and $\delta$ -functions* . . . . .	16
<b>12 Maxwell's equations</b>	<b>17</b>
12.1 Laws of electromagnetism . . . . .	17
12.2 Static charges and steady currents . . . . .	17
12.3 Electromagnetic waves . . . . .	17
<b>13 Tensors and tensor fields</b>	<b>18</b>
13.1 Definition . . . . .	18
13.2 Tensor algebra . . . . .	18
13.3 Symmetric and antisymmetric tensors . . . . .	18
13.4 Tensors, multi-linear maps and the quotient rule . . . . .	18
13.5 Tensor calculus . . . . .	18
<b>14 Tensors of rank 2</b>	<b>19</b>
14.1 Decomposition of a second-rank tensor . . . . .	19
14.2 The inertia tensor . . . . .	19
14.3 Diagonalization of a symmetric second rank tensor . . . . .	19
<b>15 Invariant and isotropic tensors</b>	<b>20</b>
15.1 Definitions and classification results . . . . .	20
15.2 Application to invariant integrals . . . . .	20

## 0 Introduction

# 1 Derivatives and coordinates

## 1.1 Derivative of functions

**Proposition.**

$$\mathbf{F}'(x) = F'_i(x)\mathbf{e}_i.$$

**Proposition.**

$$\begin{aligned}\frac{d}{dt}(f\mathbf{g}) &= \frac{df}{dt}\mathbf{g} + f\frac{d\mathbf{g}}{dt} \\ \frac{d}{dt}(\mathbf{g} \cdot \mathbf{h}) &= \frac{d\mathbf{g}}{dt} \cdot \mathbf{h} + \mathbf{g} \cdot \frac{d\mathbf{h}}{dt} \\ \frac{d}{dt}(\mathbf{g} \times \mathbf{h}) &= \frac{d\mathbf{g}}{dt} \times \mathbf{h} + \mathbf{g} \times \frac{d\mathbf{h}}{dt}\end{aligned}$$

Note that the order of multiplication must be retained in the case of the cross product.

**Theorem.** The gradient is

$$\nabla f = \frac{\partial f}{\partial x_i}\mathbf{e}_i$$

**Theorem** (Chain rule). Given a function  $f(\mathbf{r}(u))$ ,

$$\frac{df}{du} = \nabla f \cdot \frac{d\mathbf{r}}{du} = \frac{\partial f}{\partial x_i} \frac{dx_i}{du}.$$

**Theorem.** The derivative of  $\mathbf{F}$  is given by

$$M_{ji} = \frac{\partial y_j}{\partial x_i}.$$

**Theorem** (Chain rule). Suppose  $g : \mathbb{R}^p \rightarrow \mathbb{R}^n$  and  $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ . Suppose that the coordinates of the vectors in  $\mathbb{R}^p, \mathbb{R}^n$  and  $\mathbb{R}^m$  are  $u_a, x_i$  and  $y_r$  respectively. By the chain rule,

$$\frac{\partial y_r}{\partial u_a} = \frac{\partial y_r}{\partial x_i} \frac{\partial x_i}{\partial u_a},$$

with summation implied. Writing in matrix form,

$$M(f \circ g)_{ra} = M(f)_{ri}M(g)_{ia}.$$

Alternatively, in operator form,

$$\frac{\partial}{\partial u_a} = \frac{\partial x_i}{\partial u_a} \frac{\partial}{\partial x_i}.$$

## 1.2 Inverse functions

## 1.3 Coordinate systems

## 2 Curves and Line

### 2.1 Parametrised curves, lengths and arc length

**Proposition.** Let  $s$  denote the arclength of a curve  $\mathbf{r}(u)$ . Then

$$\frac{ds}{du} = \pm \left| \frac{d\mathbf{r}}{du} \right| = \pm |\mathbf{r}'(u)|$$

with the sign depending on whether it is in the direction of increasing or decreasing arclength.

**Proposition.**  $ds = \pm |\mathbf{r}'(u)| du$

### 2.2 Line integrals of vector fields

### 2.3 Gradients and Differentials

**Theorem.** If  $\mathbf{F} = \nabla f(\mathbf{r})$ , then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(\mathbf{b}) - f(\mathbf{a}),$$

where  $\mathbf{b}$  and  $\mathbf{a}$  are the end points of the curve.

In particular, the line integral does *not* depend on the curve, but the end points only. This is the vector counterpart of the fundamental theorem of calculus. A special case is when  $C$  is a closed curve, then  $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ .

**Proposition.** If  $\mathbf{F} = \nabla f$  for some  $f$ , then

$$\frac{\partial F_i}{\partial x_j} = \frac{\partial F_j}{\partial x_i}.$$

This is because both are equal to  $\partial^2 f / \partial x_i \partial x_j$ .

**Proposition.**

$$\begin{aligned} d(\lambda f + \mu g) &= \lambda df + \mu dg \\ d(fg) &= (df)g + f(dg) \end{aligned}$$

### 2.4 Work and potential energy

### 3 Integration in $\mathbb{R}^2$ and $\mathbb{R}^3$

#### 3.1 Integrals over subsets of $\mathbb{R}^2$

**Proposition.**

$$\int_D f(x, y) \, dA = \int_Y \left( \int_{x_y} f(x, y) \, dx \right) dy.$$

with  $x_y$  ranging over  $\{x : (x, y) \in D\}$ .

**Theorem** (Fubini's theorem). If  $f$  is a continuous function and  $D$  is a compact (i.e. closed and bounded) subset of  $\mathbb{R}^2$ , then

$$\iint f \, dx \, dy = \iint f \, dy \, dx.$$

While we have rather strict conditions for this theorem, it actually holds in many more cases, but those situations have to be checked manually.

**Proposition.**  $dA = dx \, dy$  in Cartesian coordinates.

**Proposition.** Take separable  $f(x, y) = h(y)g(x)$  and  $D$  be a rectangle  $\{(x, y) : a \leq x \leq b, c \leq y \leq d\}$ . Then

$$\int_D f(x, y) \, dx \, dy = \left( \int_a^b g(x) \, dx \right) \left( \int_c^d h(y) \, dy \right)$$

#### 3.2 Change of variables for an integral in $\mathbb{R}^2$

**Proposition.** Suppose we have a change of variables  $(x, y) \leftrightarrow (u, v)$  that is smooth and invertible, with regions  $D, D'$  in one-to-one correspondence. Then

$$\int_D f(x, y) \, dx \, dy = \int_{D'} f(x(u, v), y(u, v)) |J| \, du \, dv,$$

where

$$J = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

is the Jacobian. In other words,

$$dx \, dy = |J| \, du \, dv.$$

#### 3.3 Generalization to $\mathbb{R}^3$

**Proposition.**  $dV = dx \, dy \, dz$ .

**Proposition.**

$$\int_V f \, dx \, dy \, dz = \int_V f |J| \, du \, dv \, dw,$$

with

$$J = \frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

**Proposition.** In cylindrical coordinates,

$$dV = \rho \, d\rho \, d\varphi \, dz.$$

In spherical coordinates

$$dV = r^2 \sin \theta \, dr \, d\theta \, d\varphi.$$

### 3.4 Further generalizations

**Proposition.**

$$\int_D f(x_1, x_2, \dots, x_n) \, dx_1 \, dx_2 \cdots dx_n = \int_{D'} f(\{x_i(\mathbf{u})\}) |J| \, du_1 \, du_2 \cdots du_n.$$

## 4 Surfaces and surface integrals

### 4.1 Surfaces and Normal

**Proposition.**  $\nabla f$  is the normal to the surface  $f(\mathbf{r}) = c$ .

### 4.2 Parametrized surfaces and area

**Proposition.** The *vector area element* is

$$d\mathbf{S} = \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} du dv.$$

The *scalar area element* is

$$dS = \left| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right| du dv.$$

### 4.3 Surface integral of vector fields

### 4.4 Change of variables in $\mathbb{R}^2$ and $\mathbb{R}^3$ revisited

## 5 Geometry of curves and surfaces

**Theorem** (Theorema Egregium).  $K$  is *intrinsic* to the surface  $S$ . It can be expressed in terms of lengths, angles etc. which are measured entirely on the surface. So  $K$  can be defined on an arbitrary surface without embedding it on a higher dimension surface.

**Theorem** (Gauss-Bonnet theorem).

$$\theta_1 + \theta_2 + \theta_3 = \pi + \int_D K \, dA,$$

integrating over the area of the triangle.

## 6 Div, Grad, Curl and $\nabla$

### 6.1 Div, Grad, Curl and $\nabla$

**Proposition.** Let  $f, g$  be scalar functions,  $\mathbf{F}, \mathbf{G}$  be vector functions, and  $\mu, \lambda$  be constants. Then

$$\begin{aligned}\nabla(\lambda f + \mu g) &= \lambda \nabla f + \mu \nabla g \\ \nabla \cdot (\lambda \mathbf{F} + \mu \mathbf{G}) &= \lambda \nabla \cdot \mathbf{F} + \mu \nabla \cdot \mathbf{G} \\ \nabla \times (\lambda \mathbf{F} + \mu \mathbf{G}) &= \lambda \nabla \times \mathbf{F} + \mu \nabla \times \mathbf{G}.\end{aligned}$$

**Proposition.** We have the following Leibnitz properties:

$$\begin{aligned}\nabla(fg) &= (\nabla f)g + f(\nabla g) \\ \nabla \cdot (f\mathbf{F}) &= (\nabla f) \cdot \mathbf{F} + f(\nabla \cdot \mathbf{F}) \\ \nabla \times (f\mathbf{F}) &= (\nabla f) \times \mathbf{F} + f(\nabla \times \mathbf{F}) \\ \nabla(\mathbf{F} \cdot \mathbf{G}) &= \mathbf{F} \times (\nabla \times \mathbf{G}) + \mathbf{G} \times (\nabla \times \mathbf{F}) + (\mathbf{F} \cdot \nabla)\mathbf{G} + (\mathbf{G} \cdot \nabla)\mathbf{F} \\ \nabla \times (\mathbf{F} \times \mathbf{G}) &= \mathbf{F}(\nabla \cdot \mathbf{G}) - \mathbf{G}(\nabla \cdot \mathbf{F}) + (\mathbf{G} \cdot \nabla)\mathbf{F} - (\mathbf{F} \cdot \nabla)\mathbf{G} \\ \nabla \cdot (\mathbf{F} \times \mathbf{G}) &= (\nabla \times \mathbf{F}) \cdot \mathbf{G} - \mathbf{F} \cdot (\nabla \times \mathbf{G})\end{aligned}$$

which can be proven by brute-forcing with suffix notation and summation convention.

### 6.2 Second-order derivatives

**Proposition.**

$$\begin{aligned}\nabla \times (\nabla f) &= 0 \\ \nabla \cdot (\nabla \times \mathbf{F}) &= 0\end{aligned}$$

**Proposition.** If  $\mathbf{F}$  is defined in all of  $\mathbb{R}^3$ , then

$$\nabla \times \mathbf{F} = 0 \Rightarrow \mathbf{F} = \nabla f$$

for some  $f$ .

**Proposition.** If  $\mathbf{H}$  is defined over all of  $\mathbb{R}^3$  and  $\nabla \cdot \mathbf{H} = 0$ , then  $\mathbf{H} = \nabla \times \mathbf{A}$  for some  $\mathbf{A}$ .

## 7 Integral theorems

### 7.1 Statement and examples

#### 7.1.1 Green's theorem (in the plane)

**Theorem** (Green's theorem). For smooth functions  $P(x, y)$ ,  $Q(x, y)$  and  $A$  a bounded region in the  $(x, y)$  plane with boundary  $\partial A = C$ ,

$$\int_A \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \int_C (P dx + Q dy).$$

Here  $C$  is assumed to be piecewise smooth, non-intersecting closed curve, traversed anti-clockwise.

#### 7.1.2 Stokes' theorem

**Theorem** (Stokes' theorem). For a smooth vector field  $\mathbf{F}(\mathbf{r})$ ,

$$\int_S \nabla \times \mathbf{F} \cdot d\mathbf{S} = \int_{\partial S} \mathbf{F} \cdot d\mathbf{r},$$

where  $S$  is a smooth, bounded surface and  $\partial S$  is a piecewise smooth boundary of  $S$ .

The direction of the line integral is as follows: If we walk along  $C$  with  $\mathbf{n}$  facing up, then the surface is on your left.

#### 7.1.3 Divergence/Gauss theorem

**Theorem** (Divergence/Gauss theorem). For a smooth vector field  $\mathbf{F}(\mathbf{r})$ ,

$$\int_V \nabla \cdot \mathbf{F} dV = \int_{\partial V} \mathbf{F} \cdot d\mathbf{S},$$

where  $V$  is a bounded volume with boundary  $\partial V$ , a piecewise smooth, closed surface, with outward normal  $\mathbf{n}$ .

## 7.2 Relating and proving integral theorems

**Proposition.** Stokes' theorem  $\Rightarrow$  Green's theorem

**Proposition.** Green's theorem  $\Rightarrow$  Stokes' theorem.

**Proposition.** Greens theorem  $\Leftrightarrow$  2D divergence theorem.

**Proposition.** 2D divergence theorem.

$$\int_A (\nabla \cdot \mathbf{G}) dA = \int_{C=\partial A} \mathbf{G} \cdot \mathbf{n} ds.$$

## 8 Some applications of integral theorems

### 8.1 Integral expressions for div and curl

**Proposition.**

$$(\nabla \cdot \mathbf{F})(\mathbf{r}_0) = \lim_{\text{diam}(V) \rightarrow 0} \frac{1}{\text{vol}(V)} \int_{\partial V} \mathbf{F} \cdot d\mathbf{S},$$

where the limit is taken over volumes containing the point  $\mathbf{r}_0$ .

**Proposition.**

$$\mathbf{n} \cdot (\nabla \times \mathbf{F})(\mathbf{r}_0) = \lim_{\text{diam}(A) \rightarrow 0} \frac{1}{\text{area}(A)} \int_{\partial A} \mathbf{F} \cdot d\mathbf{r},$$

where the limit is taken over all surfaces  $A$  containing  $\mathbf{r}_0$  with normal  $\mathbf{n}$ .

### 8.2 Conservative fields and scalar products

**Proposition.** If (iii)  $\nabla \times \mathbf{F} = 0$ , then (ii)  $\int_C \mathbf{F} \cdot d\mathbf{r}$  is independent of  $C$ .

**Proposition.** If (ii)  $\int_C \mathbf{F} \cdot d\mathbf{r}$  is independent of  $C$  for fixed end points and orientation, then (i)  $\mathbf{F} = \nabla f$  for some scalar field  $f$ .

### 8.3 Conservation laws

## 9 Orthogonal curvilinear coordinates

### 9.1 Line, area and volume elements

### 9.2 Grad, Div and Curl

**Proposition.**

$$\nabla f = \frac{1}{h_u} \frac{\partial f}{\partial u} \mathbf{e}_u + \frac{1}{h_v} \frac{\partial f}{\partial v} \mathbf{e}_v + \frac{1}{h_w} \frac{\partial f}{\partial w} \mathbf{e}_w.$$

**Proposition.**

$$\nabla = \frac{1}{h_u} \mathbf{e}_u \frac{\partial}{\partial u} + \frac{1}{h_v} \mathbf{e}_v \frac{\partial}{\partial v} + \frac{1}{h_w} \mathbf{e}_w \frac{\partial}{\partial w}.$$

**Proposition.**

$$\begin{aligned} \nabla \times \mathbf{F} &= \frac{1}{h_v h_w} \left[ \frac{\partial}{\partial v} (h_w F_w) - \frac{\partial}{\partial w} (h_v F_v) \right] \mathbf{e}_u + \text{two similar terms} \\ &= \frac{1}{h_u h_v h_w} \begin{vmatrix} h_u \mathbf{e}_u & h_v \mathbf{e}_v & h_w \mathbf{e}_w \\ \frac{\partial}{\partial u} & \frac{\partial}{\partial v} & \frac{\partial}{\partial w} \\ h_u F_u & h_v F_v & h_w F_w \end{vmatrix} \end{aligned}$$

and

$$\nabla \cdot \mathbf{F} = \frac{1}{h_u h_v h_w} \left[ \frac{\partial}{\partial u} (h_v h_w F_u) + \text{two similar terms} \right].$$

## 10 Gauss' Law and Poisson's equation

### 10.1 Laws of gravitation

**Law** (Gauss' law for gravitation). Given any volume  $V$  bounded by closed surface  $S$ ,

$$\int_S \mathbf{g} \cdot d\mathbf{S} = -4\pi GM,$$

where  $G$  is Newton's gravitational constant, and  $M$  is the total mass contained in  $V$ .

**Law** (Gauss' Law for gravitation in differential form).

$$\nabla \cdot \mathbf{g} = -4\pi G\rho.$$

### 10.2 Laws of electrostatics

**Law** (Gauss' law for electrostatic forces).

$$\int_S \mathbf{E} \cdot d\mathbf{S} = \frac{Q}{\varepsilon_0},$$

where  $\varepsilon_0$  is the *permittivity of free space*, or *electric constant*.

**Law** (Gauss' law for electrostatic forces in differential form).

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}.$$

### 10.3 Poisson's Equation and Laplace's equation

## 11 Laplace's and Poisson's equations

### 11.1 Uniqueness theorems

**Theorem.** Consider  $\nabla^2\varphi = -\rho$  for some  $\rho(\mathbf{r})$  on a bounded volume  $V$  with  $S = \partial V$  being a closed surface, with an outward normal  $\mathbf{n}$ .

Suppose  $\varphi$  satisfies either

- (i) Dirichlet condition,  $\varphi(\mathbf{r}) = f(\mathbf{r})$  on  $S$
- (ii) Neumann condition  $\frac{\partial\varphi(\mathbf{r})}{\partial\mathbf{n}} = \mathbf{n} \cdot \nabla\varphi = g(\mathbf{r})$  on  $S$ .

where  $f, g$  are given. Then

- (i)  $\varphi(\mathbf{r})$  is unique
- (ii)  $\varphi(\mathbf{r})$  is unique up to a constant.

**Proposition** (Green's first identity).

$$\int_S (u\nabla v) \cdot d\mathbf{S} = \int_V (\nabla u) \cdot (\nabla v) dV + \int_V u\nabla^2 v dV,$$

**Proposition** (Green's second identity).

$$\int_S (u\nabla v - v\nabla u) \cdot d\mathbf{S} = \int_V (u\nabla^2 v - v\nabla^2 u) dV.$$

### 11.2 Laplace's equation and harmonic functions

#### 11.2.1 The mean value property

**Proposition** (Mean value property). Suppose  $\varphi(\mathbf{r})$  is harmonic on region  $V$  containing a solid sphere defined by  $|\mathbf{r} - \mathbf{a}| \leq R$ , with boundary  $S_R = |\mathbf{r} - \mathbf{a}| = R$ , for some  $R$ . Define

$$\bar{\varphi}(R) = \frac{1}{4\pi R^2} \int_{S_R} \varphi(\mathbf{r}) dS.$$

Then  $\varphi(\mathbf{a}) = \bar{\varphi}(R)$ .

#### 11.2.2 The maximum (or minimum) principle

**Proposition** (Maximum principle). If a function  $\varphi$  is harmonic on a region  $V$ , then  $\varphi$  cannot have a maximum at an interior point of  $\mathbf{a}$  of  $V$ .

### 11.3 Integral solutions of Poisson's equations

#### 11.3.1 Statement and informal derivation

**Proposition.** The solution to Poisson's equation  $\nabla^2\varphi = -\rho$ , with boundary conditions  $|\varphi(\mathbf{r})| = O(1/|\mathbf{r}|)$  and  $|\nabla\varphi(\mathbf{r})| = O(1/|\mathbf{r}|^2)$ , is

$$\varphi(\mathbf{r}) = \frac{1}{4\pi} \int_{V'} \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} dV'$$

For  $\rho(\mathbf{r}')$  non-zero everywhere, but suitably well-behaved as  $|\mathbf{r}'| \rightarrow \infty$ , we can also take  $V' = \mathbb{R}^3$ .

#### 11.3.2 Point sources and $\delta$ -functions\*

## 12 Maxwell's equations

### 12.1 Laws of electromagnetism

**Law** (Lorentz force law). A point charge  $q$  experiences a force of

$$\mathbf{F} = q(\mathbf{E} + \dot{\mathbf{r}} \times \mathbf{B}).$$

**Law** (Maxwell's equations).

$$\begin{aligned}\nabla \cdot \mathbf{E} &= \frac{\rho}{\varepsilon_0} \\ \nabla \cdot \mathbf{B} &= 0 \\ \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} &= 0 \\ \nabla \times \mathbf{B} - \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} &= \mu_0 \mathbf{j},\end{aligned}$$

where  $\varepsilon_0$  is the electric constant (permittivity of free space) and  $\mu_0$  is the magnetic constant (permeability of free space), which are constants determined experimentally.

### 12.2 Static charges and steady currents

### 12.3 Electromagnetic waves

## 13 Tensors and tensor fields

### 13.1 Definition

### 13.2 Tensor algebra

### 13.3 Symmetric and antisymmetric tensors

### 13.4 Tensors, multi-linear maps and the quotient rule

**Proposition** (Quotient rule). Suppose that  $T_{i\dots jp\dots q}$  is an array defined in each coordinate system, and that  $v_{i\dots j} = T_{i\dots jp\dots q}u_{p\dots q}$  is also a tensor for any tensor  $u_{p\dots q}$ . Then  $T_{i\dots jp\dots q}$  is also a tensor.

### 13.5 Tensor calculus

**Proposition.**

$$\underbrace{\frac{\partial}{\partial x_p} \cdots \frac{\partial}{\partial x_q}}_m T_{\underbrace{ij \cdots k}_n}, \quad (*)$$

is a tensor of rank  $n + m$ .

**Theorem** (Divergence theorem for tensors).

$$\int_S T_{ij\dots k\ell} n_\ell \, dS = \int_V \frac{\partial}{\partial x_\ell} (T_{ij\dots k\ell}) \, dV,$$

with  $\mathbf{n}$  being an outward pointing normal.

## **14 Tensors of rank 2**

### **14.1 Decomposition of a second-rank tensor**

### **14.2 The inertia tensor**

### **14.3 Diagonalization of a symmetric second rank tensor**

## 15 Invariant and isotropic tensors

### 15.1 Definitions and classification results

**Theorem.**

- (i) There are no isotropic tensors of rank 1, except the zero tensor.
- (ii) The most general rank 2 isotropic tensor is  $T_{ij} = \alpha\delta_{ij}$  for some scalar  $\alpha$ .
- (iii) The most general rank 3 isotropic tensor is  $T_{ijk} = \beta\varepsilon_{ijk}$  for some scalar  $\beta$ .
- (iv) All isotropic tensors of higher rank are obtained by combining  $\delta_{ij}$  and  $\varepsilon_{ijk}$  using tensor products, contractions, and linear combinations.

### 15.2 Application to invariant integrals

**Theorem.** Let

$$T_{ij\dots k} = \int_V f(\mathbf{x})x_ix_j\dots x_k \, dV.$$

where  $f(\mathbf{x})$  is a scalar function and  $V$  is some volume.

Given a rotation  $R_{ij}$ , consider an *active* transformation:  $\mathbf{x} = x_i\mathbf{e}_i$  is mapped to  $\mathbf{x}' = x'_i\mathbf{e}_i$  with  $x'_i = R_{ij}x_j$ , i.e. we map the components but not the basis, and  $V$  is mapped to  $V'$ .

Suppose that under this active transformation,

- (i)  $f(\mathbf{x}) = f(\mathbf{x}')$ ,
- (ii)  $V' = V$  (e.g. if  $V$  is all of space or a sphere).

Then  $T_{ij\dots k}$  is invariant under the rotation.