Part IA — Vector Calculus
Theorems

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Curves in $\mathbb{R}^3$
Parameterised curves and arc length, tangents and normals to curves in $\mathbb{R}^3$, the radius of curvature. [1]

Integration in $\mathbb{R}^2$ and $\mathbb{R}^3$
Line integrals. Surface and volume integrals: definitions, examples using Cartesian, cylindrical and spherical coordinates; change of variables. [4]

Vector operators
Directional derivatives. The gradient of a real-valued function: definition; interpretation as normal to level surfaces; examples including the use of cylindrical, spherical *and general orthogonal curvilinear* coordinates.

Divergence, curl and ∇^2 in Cartesian coordinates, examples; formulae for these operators (statement only) in cylindrical, spherical *and general orthogonal curvilinear* coordinates. Solenoidal fields, irrotational fields and conservative fields; scalar potentials. Vector derivative identities. [5]

Integration theorems
Divergence theorem, Green's theorem, Stokes's theorem, Green's second theorem: statements; informal proofs; examples; application to fluid dynamics, and to electromagnetism including statement of Maxwell's equations. [5]

Laplace's equation
Laplace's equation in $\mathbb{R}^2$ and $\mathbb{R}^3$: uniqueness theorem and maximum principle. Solution of Poisson's equation by Gauss's method (for spherical and cylindrical symmetry) and as an integral. [4]

Cartesian tensors in $\mathbb{R}^3$
Tensor transformation laws, addition, multiplication, contraction, with emphasis on tensors of second rank. Isotropic second and third rank tensors. Symmetric and antisymmetric tensors. Revision of principal axes and diagonalization. Quotient theorem. Examples including inertia and conductivity. [5]

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0 Introduction

1 Derivatives and coordinates

1.1 Derivative of functions

Proposition.

$$\mathbf{F}'(x) = F_i'(x)\mathbf{e}_i.$$

Proposition.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}(f\mathbf{g}) &= \frac{\mathrm{d}f}{\mathrm{d}t}\mathbf{g} + f\frac{\mathrm{d}\mathbf{g}}{\mathrm{d}t}\\
\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{g}\cdot\mathbf{h}) &= \frac{\mathrm{d}\mathbf{g}}{\mathrm{d}t}\cdot\mathbf{h} + \mathbf{g}\cdot\frac{\mathrm{d}\mathbf{h}}{\mathrm{d}t}\\
\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{g}\times\mathbf{h}) &= \frac{\mathrm{d}\mathbf{g}}{\mathrm{d}t}\times\mathbf{h} + \mathbf{g}\times\frac{\mathrm{d}\mathbf{h}}{\mathrm{d}t}
\end{aligned}$$

Note that the order of multiplication must be retained in the case of the cross product.

Theorem. The gradient is

$$\nabla f = \frac{\partial f}{\partial x_i}\mathbf{e}_i$$

Theorem (Chain rule)**.** Given a function $f(\mathbf{r}(u))$,

$$\frac{\mathrm{d}f}{\mathrm{d}u} = \nabla f\cdot\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}u} = \frac{\partial f}{\partial x_i}\frac{\mathrm{d}x_i}{\mathrm{d}u}.$$

Theorem. The derivative of $\mathbf{F}$ is given by

$$M_{ji} = \frac{\partial y_j}{\partial x_i}.$$

Theorem (Chain rule)**.** Suppose $g:\mathbb{R}^p\to\mathbb{R}^n$ and $f:\mathbb{R}^n\to\mathbb{R}^m$. Suppose that the coordinates of the vectors in $\mathbb{R}^p, \mathbb{R}^n$ and $\mathbb{R}^m$ are u_a, x_i and y_r respectively. By the chain rule,

$$\frac{\partial y_r}{\partial u_a} = \frac{\partial y_r}{\partial x_i}\frac{\partial x_i}{\partial u_a},$$

with summation implied. Writing in matrix form,

$$M(f\circ g)_{ra} = M(f)_{ri}M(g)_{ia}.$$

Alternatively, in operator form,

$$\frac{\partial}{\partial u_a} = \frac{\partial x_i}{\partial u_a}\frac{\partial}{\partial x_i}.$$

1.2 Inverse functions

1.3 Coordinate systems

2 Curves and Line

2.1 Parametrised curves, lengths and arc length

Proposition. Let s denote the arclength of a curve $\mathbf{r}(u)$. Then

$$\frac{\mathrm{d}s}{\mathrm{d}u} = \pm\left|\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}u}\right| = \pm|\mathbf{r}'(u)|$$

with the sign depending on whether it is in the direction of increasing or decreasing arclength.

Proposition. $\mathrm{d}s = \pm|\mathbf{r}'(u)|\mathrm{d}u$

2.2 Line integrals of vector fields

2.3 Gradients and Differentials

Theorem. If $\mathbf{F} = \nabla f(\mathbf{r})$, then

$$\int_C \mathbf{F}\cdot \mathrm{d}\mathbf{r} = f(\mathbf{b}) - f(\mathbf{a}),$$

where $\mathbf{b}$ and $\mathbf{a}$ are the end points of the curve.

In particular, the line integral does *not* depend on the curve, but the end points only. This is the vector counterpart of the fundamental theorem of calculus. A special case is when C is a closed curve, then $\oint_C \mathbf{F}\cdot \mathrm{d}\mathbf{r} = 0$.

Proposition. If $\mathbf{F} = \nabla f$ for some f, then

$$\frac{\partial F_i}{\partial x_j} = \frac{\partial F_j}{\partial x_i}.$$

This is because both are equal to $\partial^2 f/\partial x_i \partial x_j$.

Proposition.

$$\begin{aligned}\mathrm{d}(\lambda f + \mu g) &= \lambda \mathrm{d}f + \mu \mathrm{d}g\\ \mathrm{d}(fg) &= (\mathrm{d}f)g + f(\mathrm{d}g)\end{aligned}$$

2.4 Work and potential energy

3 Integration in $\mathbb{R}^2$ and $\mathbb{R}^3$

3.1 Integrals over subsets of $\mathbb{R}^2$

Proposition.

$$\int_D f(x,y)\,\mathrm{d}A = \int_Y \left(\int_{x_y} f(x,y)\,\mathrm{d}x\right)\mathrm{d}y.$$

with x_y ranging over $\{x : (x,y) \in D\}$.

Theorem (Fubini's theorem)**.** If f is a continuous function and D is a compact (i.e. closed and bounded) subset of $\mathbb{R}^2$, then

$$\iint f\,\mathrm{d}x\,\mathrm{d}y = \iint f\,\mathrm{d}y\,\mathrm{d}x.$$

While we have rather strict conditions for this theorem, it actually holds in many more cases, but those situations have to be checked manually.

Proposition. $\mathrm{d}A = \mathrm{d}x\,\mathrm{d}y$ in Cartesian coordinates.

Proposition. Take separable $f(x,y) = h(y)g(x)$ and D be a rectangle $\{(x,y) : a \leq x \leq b, c \leq y \leq d\}$. Then

$$\int_D f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \left(\int_a^b g(x)\,\mathrm{d}x\right)\left(\int_c^d h(y)\,\mathrm{d}y\right)$$

3.2 Change of variables for an integral in $\mathbb{R}^2$

Proposition. Suppose we have a change of variables $(x,y) \leftrightarrow (u,v)$ that is smooth and invertible, with regions D, D' in one-to-one correspondence. Then

$$\int_D f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_{D'} f(x(u,v), y(u,v))|J|\,\mathrm{d}u\,\mathrm{d}v,$$

where

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

is the Jacobian. In other words,

$$\mathrm{d}x\,\mathrm{d}y = |J|\,\mathrm{d}u\,\mathrm{d}v.$$

3.3 Generalization to $\mathbb{R}^3$

Proposition. $\mathrm{d}V = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$.

Proposition.

$$\int_V f\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \int_V f|J|\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}w,$$

with

$$J = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

Proposition. In cylindrical coordinates,

$$\mathrm{d}V = \rho\;\mathrm{d}\rho\;\mathrm{d}\varphi\;\mathrm{d}z.$$

In spherical coordinates

$$\mathrm{d}V = r^2\sin\theta\;\mathrm{d}r\;\mathrm{d}\theta\;\mathrm{d}\varphi.$$

3.4 Further generalizations

Proposition.

$$\int_D f(x_1, x_2, \cdots x_n)\;\mathrm{d}x_1\;\mathrm{d}x_2\cdots\;\mathrm{d}x_n = \int_{D'} f(\{x_i(\mathbf{u})\})|J|\;\mathrm{d}u_1\;\mathrm{d}u_2\cdots\;\mathrm{d}u_n.$$

4 Surfaces and surface integrals

4.1 Surfaces and Normal

Proposition. ∇f is the normal to the surface $f(\mathbf{r}) = c$.

4.2 Parametrized surfaces and area

Proposition. The *vector area element* is

$$\mathrm{d}\mathbf{S} = \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\,\mathrm{d}u\,\mathrm{d}v.$$

The *scalar area element* is

$$\mathrm{d}S = \left|\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right|\,\mathrm{d}u\,\mathrm{d}v.$$

4.3 Surface integral of vector fields

4.4 Change of variables in $\mathbb{R}^2$ and $\mathbb{R}^3$ revisited

5 Geometry of curves and surfaces

Theorem (Theorema Egregium). K is *intrinsic* to the surface S. It can be expressed in terms of lengths, angles etc. which are measured entirely on the surface. So K can be defined on an arbitrary surface without embedding it on a higher dimension surface.

Theorem (Gauss-Bonnet theorem).

$$\theta_1 + \theta_2 + \theta_3 = \pi + \int_D K \; \mathrm{d}A,$$

integrating over the area of the triangle.

6 Div, Grad, Curl and ∇

6.1 Div, Grad, Curl and ∇

Proposition. Let f, g be scalar functions, $\mathbf{F}, \mathbf{G}$ be vector functions, and μ, λ be constants. Then

$$\begin{aligned}
\nabla(\lambda f + \mu g) &= \lambda \nabla f + \mu \nabla g\\
\nabla \cdot (\lambda \mathbf{F} + \mu \mathbf{G}) &= \lambda \nabla \cdot \mathbf{F} + \mu \nabla \cdot \mathbf{G}\\
\nabla \times (\lambda \mathbf{F} + \mu \mathbf{G}) &= \lambda \nabla \times \mathbf{F} + \mu \nabla \times \mathbf{G}.
\end{aligned}$$

Proposition. We have the following Leibnitz properties:

$$\begin{aligned}
\nabla(fg) &= (\nabla f)g + f(\nabla g)\\
\nabla \cdot (f\mathbf{F}) &= (\nabla f) \cdot \mathbf{F} + f(\nabla \cdot \mathbf{F})\\
\nabla \times (f\mathbf{F}) &= (\nabla f) \times \mathbf{F} + f(\nabla \times \mathbf{F})\\
\nabla(\mathbf{F} \cdot \mathbf{G}) &= \mathbf{F} \times (\nabla \times \mathbf{G}) + \mathbf{G} \times (\nabla \times \mathbf{F}) + (\mathbf{F} \cdot \nabla)\mathbf{G} + (\mathbf{G} \cdot \nabla)\mathbf{F}\\
\nabla \times (\mathbf{F} \times \mathbf{G}) &= \mathbf{F}(\nabla \cdot \mathbf{G}) - \mathbf{G}(\nabla \cdot \mathbf{F}) + (\mathbf{G} \cdot \nabla)\mathbf{F} - (\mathbf{F} \cdot \nabla)\mathbf{G}\\
\nabla \cdot (\mathbf{F} \times \mathbf{G}) &= (\nabla \times \mathbf{F}) \cdot \mathbf{G} - \mathbf{F} \cdot (\nabla \times \mathbf{G})
\end{aligned}$$

which can be proven by brute-forcing with suffix notation and summation convention.

6.2 Second-order derivatives

Proposition.

$$\begin{aligned}
\nabla \times (\nabla f) &= 0\\
\nabla \cdot (\nabla \times \mathbf{F}) &= 0
\end{aligned}$$

Proposition. If $\mathbf{F}$ is defined in all of $\mathbb{R}^3$, then

$$\nabla \times \mathbf{F} = 0 \Rightarrow \mathbf{F} = \nabla f$$

for some f.

Proposition. If $\mathbf{H}$ is defined over all of $\mathbb{R}^3$ and $\nabla \cdot \mathbf{H} = 0$, then $\mathbf{H} = \nabla \times \mathbf{A}$ for some $\mathbf{A}$.

7 Integral theorems

7.1 Statement and examples

7.1.1 Green's theorem (in the plane)

Theorem (Green's theorem)**.** For smooth functions $P(x, y)$, $Q(x, y)$ and A a bounded region in the (x, y) plane with boundary $\partial A = C$,

$$\int_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \,\mathrm{d}A = \int_C (P \,\mathrm{d}x + Q \,\mathrm{d}y).$$

Here C is assumed to be piecewise smooth, non-intersecting closed curve, traversed anti-clockwise.

7.1.2 Stokes' theorem

Theorem (Stokes' theorem)**.** For a smooth vector field $\mathbf{F}(\mathbf{r})$,

$$\int_S \nabla \times \mathbf{F} \cdot \mathrm{d}\mathbf{S} = \int_{\partial S} \mathbf{F} \cdot \mathrm{d}\mathbf{r},$$

where S is a smooth, bounded surface and ∂S is a piecewise smooth boundary of S.

The direction of the line integral is as follows: If we walk along C with $\mathbf{n}$ facing up, then the surface is on your left.

7.1.3 Divergence/Gauss theorem

Theorem (Divergence/Gauss theorem)**.** For a smooth vector field $\mathbf{F}(\mathbf{r})$,

$$\int_V \nabla \cdot \mathbf{F} \,\mathrm{d}V = \int_{\partial V} \mathbf{F} \cdot \mathrm{d}\mathbf{S},$$

where V is a bounded volume with boundary ∂V, a piecewise smooth, closed surface, with outward normal $\mathbf{n}$.

7.2 Relating and proving integral theorems

Proposition. Stokes' theorem $\Rightarrow$ Green's theorem

Proposition. Green's theorem $\Rightarrow$ Stokes' theorem.

Proposition. Greens theorem $\Leftrightarrow$ 2D divergence theorem.

Proposition. 2D divergence theorem.

$$\int_A (\nabla \cdot \mathbf{G}) \,\mathrm{d}A = \int_{C = \partial A} \mathbf{G} \cdot \mathbf{n} \,\mathrm{d}s.$$

8 Some applications of integral theorems

8.1 Integral expressions for div and curl

Proposition.

$$(\nabla \cdot \mathbf{F})(\mathbf{r}_0) = \lim_{\mathrm{diam}(V) \to 1} \frac{1}{\mathrm{vol}(V)} \int_{\partial V} \mathbf{F} \cdot \mathrm{d}\mathbf{S},$$

where the limit is taken over volumes containing the point $\mathbf{r}_0$.

Proposition.

$$\mathbf{n} \cdot (\nabla \times \mathbf{F})(\mathbf{r}_0) = \lim_{\mathrm{diam}(A) \to 0} \frac{1}{\mathrm{area}(A)} \int_{\partial A} \mathbf{F} \cdot \mathrm{d}\mathbf{r},$$

where the limit is taken over all surfaces A containing $\mathbf{r}_0$ with normal $\mathbf{n}$.

8.2 Conservative fields and scalar products

Proposition. If (iii) $\nabla \times \mathbf{F} = 0$, then (ii) $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$ is independent of C.

Proposition. If (ii) $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$ is independent of C for fixed end points and orientation, then (i) $\mathbf{F} = \nabla f$ for some scalar field f.

8.3 Conservation laws

9 Orthogonal curvilinear coordinates

9.1 Line, area and volume elements

9.2 Grad, Div and Curl

Proposition.
$$\nabla f = \frac{1}{h_u}\frac{\partial f}{\partial u}\mathbf{e}_u + \frac{1}{h_v}\frac{\partial f}{\partial v}\mathbf{e}_v + \frac{1}{h_w}\frac{\partial f}{\partial w}\mathbf{e}_w.$$

Proposition.
$$\nabla = \frac{1}{h_u}\mathbf{e}_u\frac{\partial}{\partial u} + \frac{1}{h_v}\mathbf{e}_v\frac{\partial}{\partial v} + \frac{1}{h_w}\mathbf{e}_w\frac{\partial}{\partial w}.$$

Proposition.
$$\begin{aligned}\nabla \times \mathbf{F} &= \frac{1}{h_v h_w}\left[\frac{\partial}{\partial v}(h_w F_w) - \frac{\partial}{\partial w}(h_v F_v)\right]\mathbf{e}_u + \text{ two similar terms}\\ &= \frac{1}{h_u h_v h_w}\begin{vmatrix} h_u\mathbf{e}_u & h_v\mathbf{e}_v & h_w\mathbf{e}_w\\ \frac{\partial}{\partial u} & \frac{\partial}{\partial v} & \frac{\partial}{\partial w}\\ h_u F_u & h_v F_v & h_w F_w\end{vmatrix}\end{aligned}$$

and
$$\nabla \cdot \mathbf{F} = \frac{1}{h_u h_v h_w}\left[\frac{\partial}{\partial u}(h_v h_w F_u) + \text{ two similar terms}\right].$$

10 Gauss' Law and Poisson's equation

10.1 Laws of gravitation

Law (Gauss' law for gravitation)**.** Given any volume V bounded by closed surface S,

$$\int_S \mathbf{g} \cdot \mathrm{d}\mathbf{S} = -4\pi GM,$$

where G is Newton's gravitational constant, and M is the total mass contained in V.

Law (Gauss' Law for gravitation in differential form)**.**

$$\nabla \cdot \mathbf{g} = -4\pi G\rho.$$

10.2 Laws of electrostatics

Law (Gauss' law for electrostatic forces)**.**

$$\int_S \mathbf{E} \cdot \mathrm{d}\mathbf{S} = \frac{Q}{\varepsilon_0},$$

where ε_0 is the *permittivity of free space*, or *electric constant*.

Law (Gauss' law for electrostatic forces in differential form)**.**

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}.$$

10.3 Poisson's Equation and Laplace's equation

11 Laplace's and Poisson's equations

11.1 Uniqueness theorems

Theorem. Consider $\nabla^2\varphi = -\rho$ for some $\rho(\mathbf{r})$ on a bounded volume V with $S = \partial V$ being a closed surface, with an outward normal $\mathbf{n}$.

Suppose φ satisfies either

(i) Dirichlet condition, $\varphi(\mathbf{r}) = f(\mathbf{r})$ on S

(ii) Neumann condition $\frac{\partial\varphi(\mathbf{r})}{\partial\mathbf{n}} = n \cdot \nabla\varphi = g(\mathbf{r})$ on S.

where f, g are given. Then

(i) $\varphi(\mathbf{r})$ is unique

(ii) $\varphi(\mathbf{r})$ is unique up to a constant.

Proposition (Green's first identity)**.**

$$\int_S (u\nabla v)\cdot \mathrm{d}S = \int_V (\nabla u)\cdot(\nabla v)\;\mathrm{d}V + \int_V u\nabla^2 v\;\mathrm{d}V,$$

Proposition (Green's second identity)**.**

$$\int_S (u\nabla v - v\nabla u)\cdot \mathrm{d}\mathbf{S} = \int_V (u\nabla^2 v - v\nabla^2 u)\;\mathrm{d}V.$$

11.2 Laplace's equation and harmonic functions

11.2.1 The mean value property

Proposition (Mean value property)**.** Suppose $\varphi(\mathbf{r})$ is harmonic on region V containing a solid sphere defined by $|\mathbf{r}-\mathbf{a}| \leq R$, with boundary $S_R = |\mathbf{r}-\mathbf{a}| = R$, for some R. Define

$$\bar{\varphi}(R) = \frac{1}{4\pi R^2}\int_{S_R} \varphi(\mathbf{r})\;\mathrm{d}S.$$

Then $\varphi(\mathbf{a}) = \bar{\varphi}(R)$.

11.2.2 The maximum (or minimum) principle

Proposition (Maximum principle)**.** If a function φ is harmonic on a region V, then φ cannot have a maximum at an interior point of $\mathbf{a}$ of V.

11.3 Integral solutions of Poisson's equations

11.3.1 Statement and informal derivation

Proposition. The solution to Poisson's equation $\nabla^2\varphi = -\rho$, with boundary conditions $|\varphi(\mathbf{r})| = O(1/|\mathbf{r}|)$ and $|\nabla\varphi(\mathbf{r})| = O(1/|\mathbf{r}|^2)$, is

$$\varphi(\mathbf{r}) = \frac{1}{4\pi}\int_{V'} \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\;\mathrm{d}V'$$

For $\rho(\mathbf{r}')$ non-zero everywhere, but suitably well-behaved as $|\mathbf{r}'| \to \infty$, we can also take $V' = \mathbb{R}^3$.

11.3.2 Point sources and δ-functions*

12 Maxwell's equations

12.1 Laws of electromagnetism

Law (Lorentz force law)**.** A point charge q experiences a force of

$$\mathbf{F} = q(\mathbf{E} + \dot{\mathbf{r}} \times \mathbf{B}).$$

Law (Maxwell's equations)**.**

$$\begin{aligned} \nabla \cdot \mathbf{E} &= \frac{\rho}{\varepsilon_0} \\ \nabla \cdot \mathbf{B} &= 0 \\ \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} &= 0 \\ \nabla \times \mathbf{B} - \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} &= \mu_0 \mathbf{j}, \end{aligned}$$

where ε_0 is the electric constant (permittivity of free space) and μ_0 is the magnetic constant (permeability of free space), which are constants determined experimentally.

12.2 Static charges and steady currents

12.3 Electromagnetic waves

13 Tensors and tensor fields

13.1 Definition

13.2 Tensor algebra

13.3 Symmetric and antisymmetric tensors

13.4 Tensors, multi-linear maps and the quotient rule

Proposition (Quotient rule)**.** Suppose that $T_{i\cdots jp\cdots q}$ is an array defined in each coordinate system, and that $v_{i\cdots j} = T_{i\cdots jp\cdots q}u_{p\cdots q}$ is also a tensor for any tensor $u_{p\cdots q}$. Then $T_{i\cdots jp\cdots q}$ is also a tensor.

13.5 Tensor calculus

Proposition.

$$\underbrace{\frac{\partial}{\partial x_p}\cdots\frac{\partial}{\partial x_q}}_{m} T_{\underbrace{ij\cdots k}_{n}}, \tag{$*$}$$

is a tensor of rank $n+m$.

Theorem (Divergence theorem for tensors)**.**

$$\int_S T_{ij\cdots k\ell}n_\ell \;\mathrm{d}S = \int_V \frac{\partial}{\partial x_\ell}(T_{ij\cdots k\ell})\;\mathrm{d}V,$$

with $\mathbf{n}$ being an outward pointing normal.

14 Tensors of rank 2

14.1 Decomposition of a second-rank tensor

14.2 The inertia tensor

14.3 Diagonalization of a symmetric second rank tensor

15 Invariant and isotropic tensors

15.1 Definitions and classification results

Theorem.

(i) There are no isotropic tensors of rank 1, except the zero tensor.

(ii) The most general rank 2 isotropic tensor is $T_{ij} = \alpha\delta_{ij}$ for some scalar α.

(iii) The most general rank 3 isotropic tensor is $T_{ijk} = \beta\varepsilon_{ijk}$ for some scalar β.

(iv) All isotropic tensors of higher rank are obtained by combining δ_{ij} and ε_{ijk} using tensor products, contractions, and linear combinations.

15.2 Application to invariant integrals

Theorem. Let

$$T_{ij\cdots k} = \int_V f(\mathbf{x}) x_i x_j \cdots x_k \, \mathrm{d}V.$$

where $f(\mathbf{x})$ is a scalar function and V is some volume.

Given a rotation R_{ij}, consider an *active* transformation: $\mathbf{x} = x_i\mathbf{e}_i$ is mapped to $\mathbf{x}' = x_i'\mathbf{e}_i$ with $x_i' = R_{ij}x_i$, i.e. we map the components but not the basis, and V is mapped to V'.

Suppose that under this active transformation,

(i) $f(\mathbf{x}) = f(\mathbf{x}')$,

(ii) $V' = V$ (e.g. if V is all of space or a sphere).

Then $T_{ij\cdots k}$ is invariant under the rotation.