Part IA — Vector Calculus
Definitions

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Lent 2015

These notes are not endorsed by the lecturers, and I have modified them (often significantly) after lectures. They are nowhere near accurate representations of what was actually lectured, and in particular, all errors are almost surely mine.

Curves in $\mathbb{R}^3$
Parameterised curves and arc length, tangents and normals to curves in $\mathbb{R}^3$, the radius of curvature. [1]

Integration in $\mathbb{R}^2$ and $\mathbb{R}^3$
Line integrals. Surface and volume integrals: definitions, examples using Cartesian, cylindrical and spherical coordinates; change of variables. [4]

Vector operators
Directional derivatives. The gradient of a real-valued function: definition; interpretation as normal to level surfaces; examples including the use of cylindrical, spherical *and general orthogonal curvilinear* coordinates.

Divergence, curl and ∇^2 in Cartesian coordinates, examples; formulae for these operators (statement only) in cylindrical, spherical *and general orthogonal curvilinear* coordinates. Solenoidal fields, irrotational fields and conservative fields; scalar potentials. Vector derivative identities. [5]

Integration theorems
Divergence theorem, Green's theorem, Stokes's theorem, Green's second theorem: statements; informal proofs; examples; application to fluid dynamics, and to electromagnetism including statement of Maxwell's equations. [5]

Laplace's equation
Laplace's equation in $\mathbb{R}^2$ and $\mathbb{R}^3$: uniqueness theorem and maximum principle. Solution of Poisson's equation by Gauss's method (for spherical and cylindrical symmetry) and as an integral. [4]

Cartesian tensors in $\mathbb{R}^3$
Tensor transformation laws, addition, multiplication, contraction, with emphasis on tensors of second rank. Isotropic second and third rank tensors. Symmetric and antisymmetric tensors. Revision of principal axes and diagonalization. Quotient theorem. Examples including inertia and conductivity. [5]

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0 Introduction

1 Derivatives and coordinates

1.1 Derivative of functions

Definition (Vector function)**.** A *vector function* is a function $\mathbf{F} : \mathbb{R} \to \mathbb{R}^n$.

Definition (Derivative of vector function)**.** A vector function $\mathbf{F}(x)$ is *differentiable* if

$$\delta\mathbf{F} \stackrel{\text{def}}{=} \mathbf{F}(x + \delta x) - \mathbf{F}(x) = \mathbf{F}'(x)\delta x + o(\delta x)$$

for some $\mathbf{F}'(x)$. $\mathbf{F}'(x)$ is called the *derivative* of $\mathbf{F}(x)$.

Definition. A *scalar function* is a function $f : \mathbb{R}^n \to \mathbb{R}$.

Definition (Limit of vector)**.** The *limit of vectors* is defined using the norm. So $\mathbf{v} \to \mathbf{c}$ iff $|\mathbf{v} - \mathbf{c}| \to 0$. Similarly, $f(\mathbf{r}) = o(\mathbf{r})$ means $\frac{|f(\mathbf{r})|}{|\mathbf{r}|} \to 0$ as $\mathbf{r} \to \mathbf{0}$.

Definition (Gradient of scalar function)**.** A scalar function $f(\mathbf{r})$ is *differentiable* at $\mathbf{r}$ if

$$\delta f \stackrel{\text{def}}{=} f(\mathbf{r} + \delta\mathbf{r}) - f(\mathbf{r}) = (\nabla f) \cdot \delta\mathbf{r} + o(\delta\mathbf{r})$$

for some vector ∇f, the *gradient* of f at $\mathbf{r}$.

Definition (Directional derivative)**.** The *directional derivative* of f along $\mathbf{n}$ is

$$\mathbf{n} \cdot \nabla f = \lim_{h \to 0} \frac{1}{h}[f(\mathbf{r} + h\mathbf{n}) - f(\mathbf{r})],$$

It refers to how fast f changes when we move in the direction of $\mathbf{n}$.

Definition (Vector field)**.** A *vector field* is a function $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^m$.

Definition (Derivative of vector field)**.** A vector field $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable if

$$\delta\mathbf{F} \stackrel{def}{=} \mathbf{F}(\mathbf{x} + \delta\mathbf{x}) - \mathbf{F}(\mathbf{x}) = M\delta\mathbf{x} + o(\delta\mathbf{x})$$

for some $m \times n$ matrix M. M is the *derivative* of $\mathbf{F}$.

Definition. A function is *smooth* if it can be differentiated any number of times. This requires that all partial derivatives exist and are totally symmetric in i, j and k (i.e. the differential operator is commutative).

1.2 Inverse functions

1.3 Coordinate systems

2 Curves and Line

2.1 Parametrised curves, lengths and arc length

Definition (Parametrisation of curve)**.** Given a curve C in $\mathbb{R}^n$, a *parametrisation* of it is a continuous and invertible function $\mathbf{r} : D \to \mathbb{R}^n$ for some $D \subseteq \mathbb{R}$ whose image is C.

$\mathbf{r}'(u)$ is a vector tangent to the curve at each point. A parametrization is *regular* if $\mathbf{r}'(u) \neq 0$ for all u.

Definition (Scalar line element)**.** The *scalar line element* of C is $\mathrm{d}s$.

2.2 Line integrals of vector fields

Definition (Line integral)**.** The *line integral* of a smooth vector field $\mathbf{F}(\mathbf{r})$ along a path C parametrised by $\mathbf{r}(u)$ along the direction (orientation) $\mathbf{r}(\alpha) \to \mathbf{r}(\beta)$ is

$$\int_C \mathbf{F}(\mathbf{r}) \cdot \mathrm{d}\mathbf{r} = \int_\alpha^\beta \mathbf{F}(\mathbf{r}(u)) \cdot \mathbf{r}'(u)\ \mathrm{d}u.$$

We say $\mathrm{d}\mathbf{r} = \mathbf{r}'(u)\mathrm{d}u$ is the *line element* on C. Note that the upper and lower limits of the integral are the end point and start point respectively, and β is not necessarily larger than α.

Definition (Closed curve)**.** A *closed curve* is a curve with the same start and end point. The line integral along a closed curve is (sometimes) written as $\oint$ and is (sometimes) called the *circulation* of $\mathbf{F}$ around C.

Definition (Piecewise smooth curve)**.** A *piecewise smooth curve* is a curve $C = C_1 + C_2 + \cdots + C_n$ with all C_i smooth with regular parametrisations. The line integral over a piecewise smooth C is

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_{C_1} \mathbf{F} \cdot \mathrm{d}\mathbf{r} + \int_{C_2} \mathbf{F} \cdot \mathrm{d}\mathbf{r} + \cdots + \int_{C_n} \mathbf{F} \cdot \mathrm{d}\mathbf{r}.$$

2.3 Gradients and Differentials

Definition (Conservative vector field)**.** If $\mathbf{F} = \nabla f$ for some f, the $\mathbf{F}$ is called a *conservative vector field.*

Definition (Exact differential)**.** A differential $\mathbf{F} \cdot \mathrm{d}\mathbf{r}$ is *exact* if there is an f such that $\mathbf{F} = \nabla f$. Then

$$\mathrm{d}f = \nabla f \cdot \mathrm{d}\mathbf{r} = \frac{\partial f}{\partial x_i}\mathrm{d}x_i.$$

2.4 Work and potential energy

Definition (Work and potential energy)**.** If $\mathbf{F}(\mathbf{r})$ is a force, then $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$ is the *work done* by the force along the curve C. It is the limit of a sum of terms $\mathbf{F}(\mathbf{r}) \cdot \delta\mathbf{r}$, i.e. the force along the direction of $\delta\mathbf{r}$.

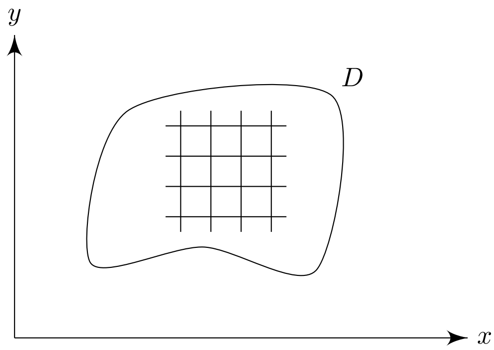
Definition (Potential energy)**.** Given a conservative force $\mathbf{F} = -\nabla V$, $V(\mathbf{x})$ is the *potential energy.* Then

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = V(\mathbf{a}) - V(\mathbf{b}).$$

3 Integration in $\mathbb{R}^2$ and $\mathbb{R}^3$

3.1 Integrals over subsets of $\mathbb{R}^2$

Definition (Surface integral)**.** Let $D \subseteq \mathbb{R}^2$. Let $\mathbf{r} = (x, y)$ be in Cartesian coordinates. We can approximate D by N disjoint subsets of simple shapes, e.g. triangles, parallelograms. These shapes are labelled by I and have areas δA_i.



To integrate a function f over D, we would like to take the sum $\sum f(\mathbf{r}_i)\delta A_i$, and take the limit as $\delta A_i \to 0$. But we need a condition stronger than simply $\delta A_i \to 0$. We won't want the areas to grow into arbitrarily long yet thin strips whose area decreases to 0. So we say that we find an ℓ such that each area can be contained in a disc of diameter ℓ.

Then we take the limit as $\ell \to 0$, $N \to \infty$, and the union of the pieces tends to D. For a function $f(\mathbf{r})$, we define the *surface integral* as

$$\int_D f(\mathbf{r})\ \mathrm{d}A = \lim_{\ell \to 0} \sum_I f(\mathbf{r}_i)\delta A_i.$$

where $\mathbf{r}_i$ is some point within each subset A_i. The integral *exists* if the limit is well-defined (i.e. the same regardless of what A_i and $\mathbf{r}_i$ we choose before we take the limit) and exists.

Definition (Area element)**.** The *area element* is $\mathrm{d}A$.

Definition (Separable function)**.** A function $f(x, y)$ is separable if it can be written as $f(x, y) = h(y)g(x)$.

3.2 Change of variables for an integral in $\mathbb{R}^2$

3.3 Generalization to $\mathbb{R}^3$

Definition (Volume integral)**.** Consider a volume $V \subseteq \mathbb{R}^3$ with position vector $\mathbf{r} = (x, y, z)$. We approximate V by N small disjoint subsets of some simple shape (e.g. cuboids) labelled by I, volume δV_I, contained within a solid sphere of diameter ℓ.

Assume that as $\ell \to 0$ and $N \to \infty$, the union of the small subsets tend to V. Then

$$\int_V f(\mathbf{r})\ \mathrm{d}V = \lim_{\ell \to 0} \sum_I f(\mathbf{r}_I^*)\delta V_I,$$

where $\mathbf{r}_I^*$ is any chosen point in each small subset.

Definition (Volume element)**.** The *volume element* is $\mathrm{d}V$.

3.4 Further generalizations

4 Surfaces and surface integrals

4.1 Surfaces and Normal

Definition (Boundary)**.** A surface S can be defined to have a *boundary* ∂S consisting of a piecewise smooth curve. If we define S as in the above examples but with the additional restriction $z \geq 0$, then ∂S is the circle $x^2 + y^2 = c$, $z = 0$.

A surface is *bounded* if it can be contained in a solid sphere, *unbounded* otherwise. A bounded surface with no boundary is called *closed* (e.g. sphere).

Definition (Orientable surface)**.** At each point, there is a unit normal $\mathbf{n}$ that's unique up to a sign.

If we can find a consistent choice of $\mathbf{n}$ that varies smoothly across S, then we say S is *orientable*, and the choice of sign of $\mathbf{n}$ is called the *orientation* of the surface.

4.2 Parametrized surfaces and area

Definition (Regular parametrization)**.** A parametrization is *regular* if for all u, v,

$$\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \neq \mathbf{0},$$

i.e. there are always two independent tangent directions.

4.3 Surface integral of vector fields

Definition (Surface integral)**.** The *surface integral* or *flux* of a vector field $\mathbf{F}(\mathbf{r})$ over S is defined by

$$\int_S \mathbf{F}(\mathbf{r}) \cdot \mathrm{d}\mathbf{S} = \int_S \mathbf{F}(\mathbf{r}) \cdot \mathbf{n}\; \mathrm{d}S = \int_D \mathbf{F}(\mathbf{r}(u, v)) \cdot \left(\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right)\; \mathrm{d}u\; \mathrm{d}v.$$

Intuitively, this is the total amount of $\mathbf{F}$ passing through S. For example, if $\mathbf{F}$ is the electric field, the flux is the amount of electric field passing through a surface.

4.4 Change of variables in $\mathbb{R}^2$ and $\mathbb{R}^3$ revisited

5 Geometry of curves and surfaces

Definition (Principal normal and curvature)**.** Write $\mathbf{t}' = \kappa\mathbf{n}$, where $\mathbf{n}$ is a unit vector and $\kappa > 0$. Then $\mathbf{n}(s)$ is called the *principal normal* and $\kappa(s)$ is called the *curvature*.

Definition (Radius of curvature)**.** The *radius of curvature* of a curve at a point $\mathbf{r}(s)$ is $1/\kappa(s)$.

Definition (Binormal)**.** The *binormal* of a curve is $\mathbf{b} = \mathbf{t} \times \mathbf{n}$.

Definition (Torsion)**.** Let $\mathbf{b}' = -\tau\mathbf{n}$. Then τ is the *torsion*.

Definition (Principal curvature)**.** The *principal curvatures* of a surface at P are the minimum and maximum possible curvature of a curve through P, denoted κ_{min} and κ_{max} respectively.

Definition (Gaussian curvature)**.** The *Gaussian curvature* of a surface at a point P is $K = \kappa_{\mathrm{min}}\kappa_{\mathrm{max}}$.

6 Div, Grad, Curl and ∇

6.1 Div, Grad, Curl and ∇

Definition (Divergence)**.** The *divergence* or *div* of $\mathbf{F}$ is

$$\nabla \cdot \mathbf{F} = \frac{\partial F_i}{\partial x_i} = \frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} + \frac{\partial F_3}{\partial x_3}.$$

Definition (Curl)**.** The *curl* of $\mathbf{F}$ is

$$\nabla \times \mathbf{F} = \varepsilon_{ijk} \frac{\partial F_k}{\partial x_j} \mathbf{e}_i = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$$

6.2 Second-order derivatives

Definition (Conservative/irrotational field and scalar potential)**.** If $\mathbf{F} = \nabla f$, then f is the *scalar potential.* We say $\mathbf{F}$ is *conservative* or *irrotational.*

Definition (Solenoidal field and vector potential)**.** If $\mathbf{H} = \nabla \times \mathbf{A}$, $\mathbf{A}$ is the *vector potential* and $\mathbf{H}$ is said to be *solenoidal.*

Definition (Laplacian operator)**.** The *Laplacian operator* is defined by

$$\nabla^2 = \nabla \cdot \nabla = \frac{\partial^2}{\partial x_i \partial x_i} = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^3} \right).$$

This operation is defined on both scalar and vector fields — on a scalar field,

$$\nabla^2 f = \nabla \cdot (\nabla f),$$

whereas on a vector field,

$$\nabla^2 \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla \times (\nabla \times \mathbf{A}).$$

7 Integral theorems

7.1 Statement and examples

7.1.1 Green's theorem (in the plane)

7.1.2 Stokes' theorem

7.1.3 Divergence/Gauss theorem

7.2 Relating and proving integral theorems

8 Some applications of integral theorems

8.1 Integral expressions for div and curl

8.2 Conservative fields and scalar products

Definition (Conservative field)**.** A vector field $\mathbf{F}$ is *conservative* if

(i) $\mathbf{F} = \nabla f$ for some scalar field f; or

(ii) $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$ is independent of C, for fixed end points and orientation; or

(iii) $\nabla \times \mathbf{F} = 0$.

In $\mathbb{R}^3$, all three formulations are equivalent.

8.3 Conservation laws

Definition (Conservation equation)**.** Suppose we are interested in a quantity Q. Let $\rho(\mathbf{r}, t)$ be the amount of stuff per unit volume and $\mathbf{j}(\mathbf{r}, t)$ be the flow rate of the quantity (eg if Q is charge, j is the current density).

The conservation equation is

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = 0.$$

9 Orthogonal curvilinear coordinates

9.1 Line, area and volume elements

Definition (Orthogonal curvilinear coordinates). u, v, w are *orthogonal curvilinear* if the tangent vectors are orthogonal.

9.2 Grad, Div and Curl

10 Gauss' Law and Poisson's equation

10.1 Laws of gravitation

Definition (Gravitational field)**.** $\mathbf{g}(\mathbf{r})$ is the *gravitational field*, *acceleration due to gravity*, or *force per unit mass*.

10.2 Laws of electrostatics

Definition (Electric field)**.** The force produced by electric charges on another charge q is $\mathbf{F} = q\mathbf{E}(\mathbf{r})$, where $\mathbf{E}(\mathbf{r})$ is the *electric field*, or force per unit charge.

Definition (Electrostatic potential)**.** If we write $\mathbf{E} = -\nabla\varphi$, then φ is the *electrostatic potential*, and

$$\nabla^2\varphi = \frac{\rho}{\varepsilon_0}.$$

10.3 Poisson's Equation and Laplace's equation

Definition (Poisson's equation)**.** The *Poisson's equation* is

$$\nabla^2\varphi = -\rho,$$

where ρ is given and $\varphi(\mathbf{r})$ is to be solved.

Definition (Laplace's equation)**.** Laplace's equation is

$$\nabla^2\varphi = 0.$$

11 Laplace's and Poisson's equations

11.1 Uniqueness theorems

11.2 Laplace's equation and harmonic functions

Definition (Harmonic function)**.** A *harmonic function* is a solution to Laplace's equation $\nabla^2\varphi = 0$.

11.2.1 The mean value property

11.2.2 The maximum (or minimum) principle

Definition (Local maximum)**.** We say that $\varphi(\mathbf{r})$ has a *local maximum* at $\mathbf{a}$ if for some $\varepsilon > 0$, $\varphi(\mathbf{r}) < \varphi(\mathbf{a})$ when $0 < |\mathbf{r} - \mathbf{a}| < \varepsilon$.

11.3 Integral solutions of Poisson's equations

11.3.1 Statement and informal derivation

11.3.2 Point sources and δ-functions*

12 Maxwell's equations

12.1 Laws of electromagnetism

Definition (Charge and current density)**.** $\rho(\mathbf{r}, t)$ is the *charge density*, defined as the charge per unit volume.

$\mathbf{j}(\mathbf{r}, t)$ is the *current density*, defined as the electric current per unit area of cross section.

12.2 Static charges and steady currents

12.3 Electromagnetic waves

13 Tensors and tensor fields

13.1 Definition

Definition (Tensor)**.** A *tensor* of rank n has components $T_{ij\cdots k}$ (with n indices) with respect to each basis $\{\mathbf{e}_i\}$ or coordinate system $\{x_i\}$, and satisfies the following rule of change of basis:

$$T'_{ij\cdots k} = R_{ip}R_{jq}\cdots R_{kr}T_{pq\cdots r}.$$

13.2 Tensor algebra

Definition (Tensor addition)**.** Tensors T and S of the same rank can be *added*; $T + S$ is also a tensor of the same rank, defined as

$$(T + S)_{ij\cdots k} = T_{ij\cdots k} + S_{ij\cdots k}.$$

in any coordinate system.

Definition (Scalar multiplication)**.** A tensor T of rank n can be multiplied by a scalar α. αT is a tensor of the same rank, defined by

$$(\alpha T)_{ij} = \alpha T_{ij}.$$

Definition (Tensor product)**.** Let T be a tensor of rank n and S be a tensor of rank m. The *tensor product* $T \otimes S$ is a tensor of rank $n + m$ defined by

$$(T \otimes S)_{x_1x_2\cdots x_ny_1y_2\cdots y_m} = T_{x_1x_2\cdots x_n}S_{y_1y_2\cdots y_n}.$$

It is trivial to show that this is a tensor.

We can similarly define tensor products for any (positive integer) number of tensors, e.g. for n vectors $\mathbf{u}, \mathbf{v}\cdots, \mathbf{w}$, we can define

$$T = \mathbf{u} \otimes \mathbf{v} \otimes \cdots \otimes \mathbf{w}$$

by

$$T_{ij\cdots k} = u_iv_j\cdots w_k,$$

as defined in the example in the beginning of the chapter.

Definition (Tensor contraction)**.** For a tensor T of rank n with components $T_{ijp\cdots q}$, we can *contract on* the indices i, j to obtain a new tensor of rank $n - 2$:

$$S_{p\cdots q} = \delta_{ij}T_{ijp\cdots q} = T_{iip\cdots q}$$

Note that we don't have to always contract on the first two indices. We can contract any pair we like.

13.3 Symmetric and antisymmetric tensors

Definition (Symmetric and anti-symmetric tensors)**.** A tensor T of rank n is *symmetric* in the indices i, j if it obeys

$$T_{ijp\cdots q} = T_{jip\cdots q}.$$

It is *anti-symmetric* if

$$T_{ijp\cdots q} = -T_{jip\cdots q}.$$

Again, a tensor can be symmetric or anti-symmetric in any pair of indices, not just the first two.

Definition (Totally symmetric and anti-symmetric tensors)**.** A tensor is *totally (anti-)symmetric* if it is (anti-)symmetric in every pair of indices.

13.4 Tensors, multi-linear maps and the quotient rule

Definition (Multilinear map)**.** A map T that maps n vectors $\mathbf{a}, \mathbf{b}, \cdots, \mathbf{c}$ to $\mathbb{R}$ is multi-linear if it is linear in each of the vectors $\mathbf{a}, \mathbf{b}, \cdots, \mathbf{c}$ individually.

13.5 Tensor calculus

Definition (Tensor field)**.** A *tensor field* is a tensor at each point in space $T_{ij\cdots k}(\mathbf{x})$, which can also be written as $T_{ij\cdots k}(x_\ell)$.

14 Tensors of rank 2

14.1 Decomposition of a second-rank tensor

14.2 The inertia tensor

Definition (Inertia tensor)**.** The *inertia tensor* is

$$I_{ij} = \sum_\alpha m_\alpha [|\mathbf{r}_\alpha|^2 \delta_{ij} - (\mathbf{r}_\alpha)_i (\mathbf{r}_\alpha)_j].$$

14.3 Diagonalization of a symmetric second rank tensor

15 Invariant and isotropic tensors

15.1 Definitions and classification results

Definition (Invariant and isotropic tensor)**.** A tensor T is *invariant* under a particular rotation R if

$$T'_{ij\cdots k} = R_{ip}R_{jq}\cdots R_{kr}T_{pq\cdots r} = T_{ij\cdots k},$$

i.e. every component is unchanged under the rotation.

A tensor T which is invariant under every rotation is *isotropic*, i.e. the same in every direction.

15.2 Application to invariant integrals