Part IA — Probability

Definitions

Based on lectures by R. Weber

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Basic concepts
Classical probability, equally likely outcomes. Combinatorial analysis, permutations and combinations. Stirling's formula (asymptotics for $\log n!$ proved). [3]

Axiomatic approach
Axioms (countable case). Probability spaces. Inclusion-exclusion formula. Continuity and subadditivity of probability measures. Independence. Binomial, Poisson and geometric distributions. Relation between Poisson and binomial distributions. Conditional probability, Bayes's formula. Examples, including Simpson's paradox. [5]

Discrete random variables
Expectation. Functions of a random variable, indicator function, variance, standard deviation. Covariance, independence of random variables. Generating functions: sums of independent random variables, random sum formula, moments.

Conditional expectation. Random walks: gambler's ruin, recurrence relations. Difference equations and their solution. Mean time to absorption. Branching processes: generating functions and extinction probability. Combinatorial applications of generating functions. [7]

Continuous random variables
Distributions and density functions. Expectations; expectation of a function of a random variable. Uniform, normal and exponential random variables. Memoryless property of exponential distribution. Joint distributions: transformation of random variables (including Jacobians), examples. Simulation: generating continuous random variables, independent normal random variables. Geometrical probability: Bertrand's paradox, Buffon's needle. Correlation coefficient, bivariate normal random variables. [6]

Inequalities and limits
Markov's inequality, Chebyshev's inequality. Weak law of large numbers. Convexity: Jensen's inequality for general random variables, AM/GM inequality.

Moment generating functions and statement (no proof) of continuity theorem. Statement of central limit theorem and sketch of proof. Examples, including sampling. [3]

Contents

0 Introduction

1 Classical probability

1.1 Classical probability

Definition (Classical probability)**.** *Classical probability* applies in a situation when there are a finite number of equally likely outcome.

Definition (Sample space)**.** The set of all possible outcomes is the *sample space*, Ω. We can lists the outcomes as $\omega_1, \omega_2, \cdots \in \Omega$. Each $\omega \in \Omega$ is an *outcome*.

Definition (Event)**.** A subset of Ω is called an *event*.

Definition (Set notations)**.** Given any two events $A, B \subseteq \Omega$,

- The *complement* of A is $A^C = A' = \bar{A} = \Omega \setminus A$.
- "A or B" is the set $A \cup B$.
- "A and B" is the set $A \cap B$.
- A and B are *mutually exclusive* or *disjoint* if $A \cap B = \emptyset$.
- If $A \subseteq B$, then A occurring implies B occurring.

Definition (Probability)**.** Suppose $\Omega = \{\omega_1, \omega_2, \cdots, \omega_N\}$. Let $A \subseteq \Omega$ be an event. Then the *probability* of A is

$$\mathbb{P}(A) = \frac{\text{Number of outcomes in } A}{\text{Number of outcomes in } \Omega} = \frac{|A|}{N}.$$

1.2 Counting

Definition (Sampling with replacement)**.** When we *sample with replacement*, after choosing at item, it is put back and can be chosen again. Then *any* sampling function f satisfies sampling with replacement.

Definition (Sampling without replacement)**.** When we *sample without replacement*, after choosing an item, we kill it with fire and cannot choose it again. Then f must be an injective function, and clearly we must have $x \geq n$.

Definition (Multinomial coefficient)**.** A *multinomial coefficient* is

$$\binom{n}{n_1, n_2, \cdots, n_k} = \binom{n}{n_1}\binom{n - n_1}{n_2} \cdots \binom{n - n_1 \cdots - n_{k-1}}{n_k} = \frac{n!}{n_1! n_2! \cdots n_k!}.$$

It is the number of ways to distribute n items into k positions, in which the ith position has n_i items.

1.3 Stirling's formula

2 Axioms of probability

2.1 Axioms and definitions

Definition (Probability space)**.** A *probability space* is a triple $(\Omega, \mathcal{F}, \mathbb{P})$. Ω is a set called the *sample space*, $\mathcal{F}$ is a collection of subsets of Ω, and $\mathbb{P} : \mathcal{F} \to [0, 1]$ is the *probability measure*.

$\mathcal{F}$ has to satisfy the following axioms:

(i) $\emptyset, \Omega \in \mathcal{F}$.

(ii) $A \in \mathcal{F} \Rightarrow A^C \in \mathcal{F}$.

(iii) $A_1, A_2, \cdots \in \mathcal{F} \Rightarrow \bigcup_{i=1}^\infty A_i \in \mathcal{F}$.

And $\mathbb{P}$ has to satisfy the following *Kolmogorov axioms*:

(i) $0 \le \mathbb{P}(A) \le 1$ for all $A \in \mathcal{F}$

(ii) $\mathbb{P}(\Omega) = 1$

(iii) For any countable collection of events $A_1, A_2, \cdots$ which are disjoint, i.e. $A_i \cap A_j = \emptyset$ for all i, j, we have

$$\mathbb{P}\left(\bigcup_i A_i\right) = \sum_i \mathbb{P}(A_i).$$

Items in Ω are known as the *outcomes*, items in $\mathcal{F}$ are known as the *events*, and $\mathbb{P}(A)$ is the *probability* of the event A.

Definition (Probability distribution)**.** Let $\Omega = \{\omega_1, \omega_2, \cdots\}$. Choose numbers $p_1, p_2, \cdots$ such that $\sum_{i=1}^\infty p_i = 1$. Let $p(\omega_i) = p_i$. Then define

$$\mathbb{P}(A) = \sum_{\omega_i \in A} p(\omega_i).$$

This $\mathbb{P}(A)$ satisfies the above axioms, and $p_1, p_2, \cdots$ is the *probability distribution*

Definition (Limit of events)**.** A sequence of events $A_1, A_2, \cdots$ is *increasing* if $A_1 \subseteq A_2 \cdots$. Then we define the *limit* as

$$\lim_{n \to \infty} A_n = \bigcup_1^\infty A_n.$$

Similarly, if they are *decreasing*, i.e. $A_1 \supseteq A_2 \cdots$, then

$$\lim_{n \to \infty} A_n = \bigcap_1^\infty A_n.$$

2.2 Inequalities and formulae

2.3 Independence

Definition (Independent events)**.** Two events A and B are *independent* if

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B).$$

Otherwise, they are said to be *dependent*.

Definition (Independence of multiple events)**.** Events $A_1, A_2, \cdots$ are said to be *mutually independent* if

$$\mathbb{P}(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_r}) = \mathbb{P}(A_{i_1})\mathbb{P}(A_{i_2}) \cdots \mathbb{P}(A_{i_r})$$

for any $i_1, i_2, \cdots i_r$ and $r \geq 2$.

2.4 Important discrete distributions

Definition (Bernoulli distribution)**.** Suppose we toss a coin. $\Omega = \{H, T\}$ and $p \in [0, 1]$. The *Bernoulli distribution*, denoted $B(1, p)$ has

$$\mathbb{P}(H) = p; \quad \mathbb{P}(T) = 1 - p.$$

Definition (Binomial distribution)**.** Suppose we toss a coin n times, each with probability p of getting heads. Then

$$\mathbb{P}(HHTT \cdots T) = pp(1 - p) \cdots (1 - p).$$

So

$$\mathbb{P}(\text{two heads}) = \binom{n}{2} p^2 (1 - p)^{n-2}.$$

In general,

$$\mathbb{P}(k \text{ heads}) = \binom{n}{k} p^k (1 - p)^{n-k}.$$

We call this the *binomial distribution* and write it as $B(n, p)$.

Definition (Geometric distribution)**.** Suppose we toss a coin with probability p of getting heads. The probability of having a head after k consecutive tails is

$$p_k = (1 - p)^k p$$

This is *geometric distribution*. We say it is *memoryless* because how many tails we've got in the past does not give us any information to how long I'll have to wait until I get a head.

Definition (Hypergeometric distribution)**.** Suppose we have an urn with n_1 red balls and n_2 black balls. We choose n balls. The probability that there are k red balls is

$$\mathbb{P}(k \text{ red}) = \frac{\binom{n_1}{k}\binom{n_2}{n-k}}{\binom{n_1+n_2}{n}}.$$

Definition (Poisson distribution)**.** The *Poisson distribution* denoted $P(\lambda)$ is

$$p_k = \frac{\lambda^k}{k!} e^{-\lambda}$$

for $k \in \mathbb{N}$.

2.5 Conditional probability

Definition (Conditional probability)**.** Suppose B is an event with $\mathbb{P}(B) > 0$. For any event $A \subseteq \Omega$, the *conditional probability of A given B* is

$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}.$$

We interpret as the probability of A happening given that B has happened.

Definition (Partition)**.** A *partition of the sample space* is a collection of disjoint events $\{B_i\}_{i=0}^{\infty}$ such that $\bigcup_i B_i = \Omega$.

3 Discrete random variables

3.1 Discrete random variables

Definition (Random variable)**.** A *random variable* X taking values in a set Ω_X is a function $X : \Omega \to \Omega_X$. Ω_X is usually a set of numbers, e.g. $\mathbb{R}$ or $\mathbb{N}$.

Definition (Discrete random variables)**.** A random variable is *discrete* if Ω_X is finite or countably infinite.

Notation. Let $T \subseteq \Omega_X$, define

$$\mathbb{P}(X \in T) = \mathbb{P}(\{\omega \in \Omega : X(\omega) \in T\}).$$

i.e. the probability that the outcome is in T.

Definition (Discrete uniform distribution)**.** A *discrete uniform distribution* is a discrete distribution with finitely many possible outcomes, in which each outcome is equally likely.

Notation. We write

$$\mathbb{P}_X(x) = \mathbb{P}(X = x).$$

We can also write $X \sim B(n, p)$ to mean

$$\mathbb{P}(X = r) = \binom{n}{r} p^r (1-p)^{n-r},$$

and similarly for the other distributions we have come up with before.

Definition (Expectation)**.** The *expectation* (or *mean*) of a real-valued X is equal to

$$\mathbb{E}[X] = \sum_{\omega \in \Omega} p_\omega X(\omega).$$

provided this is *absolutely convergent*. Otherwise, we say the expectation doesn't exist. Alternatively,

$$\begin{aligned}
\mathbb{E}[X] &= \sum_{x \in \Omega_X} \sum_{\omega : X(\omega) = x} p_\omega X(\omega) \\
&= \sum_{x \in \Omega_X} x \sum_{\omega : X(\omega) = x} p_\omega \\
&= \sum_{x \in \Omega_X} x P(X = x).
\end{aligned}$$

We are sometimes lazy and just write $\mathbb{E}X$.

Definition (Variance and standard deviation)**.** The *variance* of a random variable X is defined as

$$\operatorname{var}(X) = \mathbb{E}[(X - \mathbb{E}[X])^2].$$

The *standard deviation* is the square root of the variance, $\sqrt{\operatorname{var}(X)}$.

Definition (Indicator function)**.** The *indicator function* or *indicator variable* $I[A]$ (or I_A) of an event $A \subseteq \Omega$ is

$$I[A](\omega) = \begin{cases} 1 & \omega \in A \\ 0 & \omega \notin A \end{cases}$$

Definition (Independent random variables)**.** Let $X_1, X_2, \cdots, X_n$ be discrete random variables. They are *independent* iff for any $x_1, x_2, \cdots, x_n$,

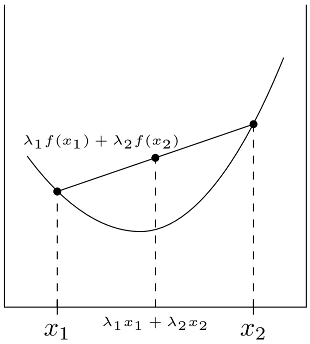
$$\mathbb{P}(X_1 = x_1, \cdots, X_n = x_n) = \mathbb{P}(X_1 = x_1) \cdots \mathbb{P}(X_n = x_n).$$

3.2 Inequalities

Definition (Convex function)**.** A function $f : (a, b) \to \mathbb{R}$ is *convex* if for all $x_1, x_2 \in (a, b)$ and $\lambda_1, \lambda_2 \geq 0$ such that $\lambda_1 + \lambda_2 = 1$,

$$\lambda_1 f(x_1) + \lambda_2 f(x_2) \geq f(\lambda_1 x_1 + \lambda_2 x_2).$$

It is *strictly convex* if the inequality above is strict (except when $x_1 = x_2$ or λ_1 or $\lambda_2 = 0$).



A function is *concave* if $-f$ is convex.

3.3 Weak law of large numbers

3.4 Multiple random variables

Definition (Covariance)**.** Given two random variables X, Y, the *covariance* is

$$\operatorname{cov}(X, Y) = \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])].$$

Definition (Correlation coefficient)**.** The *correlation coefficient* of X and Y is

$$\operatorname{corr}(X, Y) = \frac{\operatorname{cov}(X, Y)}{\sqrt{\operatorname{var}(X)\operatorname{var}(Y)}}.$$

Definition (Conditional distribution)**.** Let X and Y be random variables (in general not independent) with joint distribution $\mathbb{P}(X = x, Y = y)$. Then the *marginal distribution* (or simply *distribution*) of X is

$$\mathbb{P}(X = x) = \sum_{y \in \Omega_y} \mathbb{P}(X = x, Y = y).$$

The *conditional distribution* of X given Y is

$$\mathbb{P}(X = x \mid Y = y) = \frac{\mathbb{P}(X = x, Y = y)}{\mathbb{P}(Y = y)}.$$

The *conditional expectation* of X given Y is

$$\mathbb{E}[X \mid Y = y] = \sum_{x \in \Omega_X} x\mathbb{P}(X = x \mid Y = y).$$

We can view $\mathbb{E}[X \mid Y]$ as a random variable in Y: given a value of Y, we return the expectation of X.

3.5 Probability generating functions

Definition (Probability generating function (pgf))**.** The *probability generating function (pgf)* of X is

$$p(z) = \mathbb{E}[z^X] = \sum_{r=0}^{\infty} \mathbb{P}(X = r)z^r = p_0 + p_1 z + p_2 z^2 \cdots = \sum_0^\infty p_r z^r.$$

This is a power series (or polynomial), and converges if $|z| \leq 1$, since

$$|p(z)| \leq \sum_r p_r |z^r| \leq \sum_r p_r = 1.$$

We sometimes write as $p_X(z)$ to indicate what the random variable.

4 Interesting problems

4.1 Branching processes

4.2 Random walk and gambler's ruin

Definition (Random walk)**.** Let $X_1, \cdots, X_n$ be iid random variables such that $X_n = +1$ with probability p, and -1 with probability $1 - p$. Let $S_n = S_0 + X_1 + \cdots + X_n$. Then $(S_0, S_1, \cdots, S_n)$ is a *1-dimensional random walk.*

If $p = q = \frac{1}{2}$, we say it is a *symmetric random walk.*

5 Continuous random variables

5.1 Continuous random variables

Definition (Continuous random variable)**.** A random variable $X : \Omega \to \mathbb{R}$ is *continuous* if there is a function $f : \mathbb{R} \to \mathbb{R}_{\geq 0}$ such that

$$\mathbb{P}(a \leq X \leq b) = \int_a^b f(x)\;\mathrm{d}x.$$

We call f the *probability density function*, which satisfies

- $f \geq 0$
- $\int_{-\infty}^{\infty} f(x) = 1$.

Definition (Cumulative distribution function)**.** The *cumulative distribution function* (or simply *distribution function*) of a random variable X (discrete, continuous, or neither) is

$$F(x) = \mathbb{P}(X \leq x).$$

Definition (Uniform distribution)**.** The *uniform distribution* on $[a, b]$ has pdf

$$f(x) = \frac{1}{b-a}.$$

Then

$$F(x) = \int_a^x f(z)\;\mathrm{d}z = \frac{x-a}{b-a}$$

for $a \leq x \leq b$.

If X follows a uniform distribution on $[a, b]$, we write $X \sim U[a, b]$.

Definition (Exponential random variable)**.** The *exponential random variable with parameter* λ has pdf

$$f(x) = \lambda e^{-\lambda x}$$

and

$$F(x) = 1 - e^{-\lambda x}$$

for $x \geq 0$.

We write $X \sim \mathcal{E}(\lambda)$.

Definition (Expectation)**.** The *expectation* (or *mean*) of a continuous random variable is

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} xf(x)\;\mathrm{d}x,$$

provided not both $\int_0^\infty xf(x)\;\mathrm{d}x$ and $\int_{-\infty}^0 xf(x)\;\mathrm{d}x$ are infinite.

Definition (Variance)**.** The *variance* of a continuous random variable is

$$\mathrm{var}(X) = \mathbb{E}[(X - \mathbb{E}[X])^2] = \mathbb{E}[X^2] - (E[X])^2.$$

Definition (Mode and median)**.** Given a pdf $f(x)$, we call $\hat{x}$ a *mode* if

$$f(\hat{x}) \geq f(x)$$

for all x. Note that a distribution can have many modes. For example, in the uniform distribution, all x are modes.

We say it is a median if

$$\int_{-\infty}^{\hat{x}} f(x)\,\mathrm{d}x = \frac{1}{2} = \int_{\hat{x}}^{\infty} f(x)\,\mathrm{d}x.$$

For a discrete random variable, the median is $\hat{x}$ such that

$$\mathbb{P}(X \leq \hat{x}) \geq \frac{1}{2},\quad \mathbb{P}(X \geq \hat{x}) \geq \frac{1}{2}.$$

Here we have a non-strict inequality since if the random variable, say, always takes value 0, then both probabilities will be 1.

Definition (Sample mean)**.** If $X_1, \cdots, X_n$ is a random sample from some distribution, then the *sample mean* is

$$\bar{X} = \frac{1}{n}\sum_1^n X_i.$$

5.2 Stochastic ordering and inspection paradox

Definition (Stochastic order)**.** The *stochastic order* is defined as: $X \geq_{\mathrm{st}} Y$ if $\mathbb{P}(X > t) \geq \mathbb{P}(Y > t)$ for all t.

5.3 Jointly distributed random variables

Definition (Joint distribution)**.** Two random variables X, Y have *joint distribution* $F : \mathbb{R}^2 \mapsto [0, 1]$ defined by

$$F(x, y) = \mathbb{P}(X \leq x, Y \leq y).$$

The *marginal distribution* of X is

$$F_X(x) = \mathbb{P}(X \leq x) = \mathbb{P}(X \leq x, Y < \infty) = F(x, \infty) = \lim_{y\to\infty} F(x, y)$$

Definition (Jointly distributed random variables)**.** We say $X_1, \cdots, X_n$ are *jointly distributed continuous random variables* and have *joint pdf* f if for any set $A \subseteq \mathbb{R}^n$

$$\mathbb{P}((X_1, \cdots, X_n) \in A) = \int_{(x_1,\cdots x_n)\in A} f(x_1, \cdots, x_n)\,\mathrm{d}x_1 \cdots \mathrm{d}x_n.$$

where

$$f(x_1, \cdots, x_n) \geq 0$$

and

$$\int_{\mathbb{R}^n} f(x_1, \cdots, x_n)\,\mathrm{d}x_1 \cdots \mathrm{d}x_n = 1.$$

Definition (Independent continuous random variables)**.** Continuous random variables $X_1, \cdots, X_n$ are independent if

$$\mathbb{P}(X_1 \in A_1, X_2 \in A_2, \cdots, X_n \in A_n) = \mathbb{P}(X_1 \in A_1)\mathbb{P}(X_2 \in A_2) \cdots \mathbb{P}(X_n \in A_n)$$

for all $A_i \subseteq \Omega_{X_i}$.

If we let F_{X_i} and f_{X_i} be the cdf, pdf of X, then

$$F(x_1, \cdots, x_n) = F_{X_1}(x_1) \cdots F_{X_n}(x_n)$$

and

$$f(x_1, \cdots, x_n) = f_{X_1}(x_1) \cdots f_{X_n}(x_n)$$

are each individually equivalent to the definition above.

5.4 Geometric probability

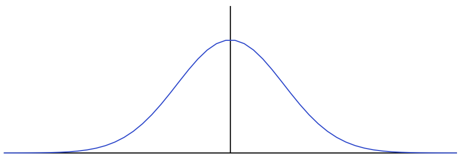
5.5 The normal distribution

Definition (Normal distribution)**.** The *normal distribution* with parameters μ, σ^2, written $N(\mu, \sigma^2)$ has pdf

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right),$$

for $-\infty < x < \infty$.

It looks approximately like this:



The *standard normal* is when $\mu = 0, \sigma^2 = 1$, i.e. $X \sim N(0, 1)$.

We usually write $\phi(x)$ for the pdf and $\Phi(x)$ for the cdf of the standard normal.

5.6 Transformation of random variables

Definition (Jacobian determinant)**.** Suppose $\frac{\partial s_i}{\partial y_j}$ exists and is continuous at every point $(y_1, \cdots, y_n) \in S$. Then the *Jacobian determinant* is

$$J = \frac{\partial(s_1, \cdots, s_n)}{\partial(y_1, \cdots, y_n)} = \det \begin{pmatrix} \frac{\partial s_1}{\partial y_1} & \cdots & \frac{\partial s_1}{\partial y_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial s_n}{\partial y_1} & \cdots & \frac{\partial s_n}{\partial y_n} \end{pmatrix}$$

Definition (Order statistics)**.** Suppose that $X_1, \cdots, X_n$ are some random variables, and $Y_1, \cdots, Y_n$ is $X_1, \cdots, X_n$ arranged in increasing order, i.e. $Y_1 \leq Y_2 \leq \cdots \leq Y_n$. This is the *order statistics*.

We sometimes write $Y_i = X_{(i)}$.

5.7 Moment generating functions

Definition (Moment generating function). The *moment generating function* of a random variable X is

$$m(\theta) = \mathbb{E}[e^{\theta X}].$$

For those θ in which $m(\theta)$ is finite, we have

$$m(\theta) = \int_{-\infty}^{\infty} e^{\theta x} f(x)\,\mathrm{d}x.$$

Definition (Moment). The rth *moment* of X is $\mathbb{E}[X^r]$.

6 More distributions

6.1 Cauchy distribution

Definition (Cauchy distribution)**.** The *Cauchy distribution* has pdf

$$f(x) = \frac{1}{\pi(1 + x^2)}$$

for $-\infty < x < \infty$.

6.2 Gamma distribution

Definition (Gamma distribution)**.** The *gamma distribution* $\Gamma(n, \lambda)$ has pdf

$$f(x) = \frac{\lambda^n x^{n-1} e^{-\lambda x}}{(n-1)!}.$$

We can show that this is a distribution by showing that it integrates to 1.

6.3 Beta distribution*

Definition (Beta distribution)**.** The *beta distribution* $\beta(a, b)$ has pdf

$$f(x; a, b) = \frac{\Gamma(a + b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1 - x)^{b-1}$$

for $0 \leq x \leq 1$.

This has mean $a/(a + b)$.

6.4 More on the normal distribution

6.5 Multivariate normal

7 Central limit theorem

8 Summary of distributions

8.1 Discrete distributions

8.2 Continuous distributions