Part IA — Dynamics and Relativity

Theorems with proof

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These notes are not endorsed by the lecturers, and I have modified them (often significantly) after lectures. They are nowhere near accurate representations of what was actually lectured, and in particular, all errors are almost surely mine.

Familiarity with the topics covered in the non-examinable Mechanics course is assumed.

Basic concepts
Space and time, frames of reference, Galilean transformations. Newton's laws. Dimensional analysis. Examples of forces, including gravity, friction and Lorentz. [4]

Newtonian dynamics of a single particle
Equation of motion in Cartesian and plane polar coordinates. Work, conservative forces and potential energy, motion and the shape of the potential energy function; stable equilibria and small oscillations; effect of damping.

Angular velocity, angular momentum, torque.

Orbits: the $u(\theta)$ equation; escape velocity; Kepler's laws; stability of orbits; motion in a repulsive potential (Rutherford scattering). Rotating frames: centrifugal and coriolis forces. *Brief discussion of Foucault pendulum.* [8]

Newtonian dynamics of systems of particles
Momentum, angular momentum, energy. Motion relative to the centre of mass; the two body problem. Variable mass problems; the rocket equation. [2]

Rigid bodies
Moments of inertia, angular momentum and energy of a rigid body. Parallel axes theorem. Simple examples of motion involving both rotation and translation (e.g. rolling). [3]

Special relativity
The principle of relativity. Relativity and simultaneity. The invariant interval. Lorentz transformations in $(1+1)$-dimensional spacetime. Time dilation and length contraction. The Minkowski metric for $(1+1)$-dimensional spacetime. Lorentz transformations in $(3+1)$ dimensions. 4-vectors and Lorentz invariants. Proper time. 4-velocity and 4-momentum. Conservation of 4-momentum in particle decay. Collisions. The Newtonian limit. [7]

Contents

0 Introduction

1 Newtonian dynamics of particles

1.1 Newton's laws of motion

Law (Newton's First Law of Motion)**.** A body remains at rest, or moves uniformly in a straight line, unless acted on by a force. (This is in fact Galileo's Law of Inertia)

Law (Newton's Second Law of Motion)**.** The rate of change of momentum of a body is equal to the force acting on it (in both magnitude and direction).

Law (Newton's Third Law of Motion)**.** To every action there is an equal and opposite reaction: the forces of two bodies on each other are equal and in opposite directions.

1.2 Galilean transformations

Law (Galilean relativity)**.** The *principle of relativity* asserts that the laws of physics are the same in inertial frames.

1.3 Newton's Second Law

Law. The *equation of motion* for a particle subject to a force $\mathbf{F}$ is

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = \mathbf{F},$$

where $\mathbf{p} = m\mathbf{v} = m\dot{\mathbf{r}}$ is the (linear) momentum of the particle. We say m is the (inertial) mass of the particle, which is a measure of its reluctance to accelerate.

2 Dimensional analysis

2.1 Units

2.2 Scaling

3 Forces

3.1 Force and potential energy in one dimension

Proposition. Suppose the equation of a particle satisfies

$$m\ddot{x} = -\frac{\mathrm{d}V}{\mathrm{d}x}.$$

Then the total energy

$$E = T + V = \frac{1}{2}m\dot{x}^2 + V(x)$$

is conserved, i.e. $\dot{E} = 0$.

Proof.

$$\begin{aligned}\frac{\mathrm{d}E}{\mathrm{d}t} &= m\dot{x}\ddot{x} + \frac{\mathrm{d}V}{\mathrm{d}x}\dot{x}\\ &= \dot{x}\left(m\ddot{x} + \frac{\mathrm{d}V}{\mathrm{d}x}\right)\\ &= 0\end{aligned}$$

□

3.2 Motion in a potential

Proposition. If $\mathbf{F}$ is conservative, then the energy

$$\begin{aligned}E &= T + V\\ &= \frac{1}{2}m|\mathbf{v}|^2 + V(\mathbf{r})\end{aligned}$$

is conserved. For any particle moving under this force, the work done is equal to the change in potential energy, and is independent of the path taken between the end points. In particular, if we travel around a closed loop, no work is done.

Proof.

$$\begin{aligned}\frac{\mathrm{d}E}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}m\dot{\mathbf{r}}\cdot\dot{\mathbf{r}} + V\right)\\ &= m\ddot{\mathbf{r}}\cdot\dot{\mathbf{r}} + \frac{\partial V}{\partial x_i}\frac{\mathrm{d}x_i}{\mathrm{d}t}\\ &= (m\ddot{\mathbf{r}} + \nabla V)\cdot\dot{\mathbf{r}}\\ &= (m\ddot{\mathbf{r}} - \mathbf{F})\cdot\dot{\mathbf{r}}\\ &= 0\end{aligned}$$

So the energy is conserved. In this case, the work done is

$$W = \int_C \mathbf{F}\cdot\mathrm{d}\mathbf{r} = -\int_C (\nabla V)\cdot\mathrm{d}\mathbf{r} = V(\mathbf{r}_1) - V(\mathbf{r}_2).$$

□

3.3 Central forces

Proposition. $\nabla r = \hat{\mathbf{r}}$.

Proof. We know that

$$r^2 = x_1^2 + x_2^2 + x_3^2.$$

Then

$$2r\frac{\partial r}{\partial x_i} = 2x_i.$$

So

$$\frac{\partial r}{\partial x_i} = \frac{x_i}{r} = (\hat{\mathbf{r}})_i. \qquad \square$$

Proposition. Let $\mathbf{F} = -\nabla V(r)$ be a central force. Then

$$\mathbf{F} = -\nabla V = -\frac{\mathrm{d}V}{\mathrm{d}r}\hat{\mathbf{r}},$$

where $\hat{\mathbf{r}} = \mathbf{r}/r$ is the unit vector in the radial direction pointing away from the origin.

Proof. Using the proof above,

$$(\nabla V)_i = \frac{\partial V}{\partial x_i} = \frac{\mathrm{d}V}{\mathrm{d}r}\frac{\partial r}{\partial x_i} = \frac{\mathrm{d}V}{\mathrm{d}r}(\hat{\mathbf{r}})_i \qquad \square$$

Proposition. Angular momentum is conserved by a central force.

Proof.

$$\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t} = m\dot{\mathbf{r}}\times\dot{\mathbf{r}} + m\mathbf{r}\times\ddot{\mathbf{r}} = \mathbf{0} + \mathbf{r}\times\mathbf{F} = \mathbf{0}.$$

where the last equality comes from the fact that $\mathbf{F}$ is parallel to $\mathbf{r}$ for a central force. $\square$

3.4 Gravity

Law (Newton's law of gravitation)**.** If a particle of mass M is fixed at a origin, then a second particle of mass m experiences a potential energy

$$V(r) = -\frac{GMm}{r},$$

where $G \approx 6.67\times 10^{-11}\,\mathrm{m}^3\,\mathrm{kg}^{-1}\,\mathrm{s}^{-2}$ is the *gravitational constant.*

The gravitational force experienced is then

$$F = -\nabla V = -\frac{GMm}{r^2}\hat{\mathbf{r}}.$$

Proposition. The external gravitational potential of a spherically symmetric object of mass M is the same as that of a point particle with the same mass at the center of the object, i.e.

$$\Phi_g(r) = -\frac{GM}{r}.$$

3.5 Electromagnetism

Law (Lorentz force law)**.** The *electromagnetic force* experienced by a particle with electric charge q is

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}).$$

Proposition. For time independent $\mathbf{E}(\mathbf{r})$ and $\mathbf{B}(\mathbf{r})$, the energy

$$E = T + V = \frac{1}{2}m|\mathbf{v}|^2 + q\Phi_e$$

is conserved.

Proof.

$$\begin{aligned}\frac{\mathrm{d}E}{\mathrm{d}t} &= m\ddot{\mathbf{r}} \cdot \dot{\mathbf{r}} + q(\nabla\Phi_e) \cdot \dot{\mathbf{r}}\\ &= (m\ddot{\mathbf{r}} - q\mathbf{E}) \cdot \dot{\mathbf{r}}\\ &= (q\dot{\mathbf{r}} \times \mathbf{B}) \cdot \dot{\mathbf{r}}\\ &= 0\end{aligned}$$

□

Law (Columb's law)**.** A particle of charge Q, fixed at the origin, produces an electrostatic potential

$$\Phi_e = \frac{Q}{4\pi\varepsilon_0 r},$$

where $\varepsilon_0 \approx 8.85 \times 10^{-12}\,\mathrm{m}^{-3}\,\mathrm{kg}^{-1}\,\mathrm{s}^2\,\mathrm{C}^2$.

The corresponding electric field is

$$\mathbf{E} = -\nabla\Phi_e = \frac{Q}{4\pi\varepsilon_0}\frac{\hat{\mathbf{r}}}{r^2}.$$

The resulting force on a particle of charge q is

$$\mathbf{F} = q\mathbf{E} = \frac{Qq}{4\pi\varepsilon_0}\frac{\hat{\mathbf{r}}}{r^2}.$$

3.6 Friction

4 Orbits

4.1 Polar coordinates in the plane

Proposition.

$$\frac{\mathrm{d}\hat{\mathbf{r}}}{\mathrm{d}\theta} = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} = \hat{\boldsymbol{\theta}}$$
$$\frac{\mathrm{d}\hat{\boldsymbol{\theta}}}{\mathrm{d}\theta} = \begin{pmatrix} -\cos\theta \\ -\sin\theta \end{pmatrix} = -\hat{\mathbf{r}}.$$

4.2 Motion in a central force field

4.3 Equation of the shape of the orbit

Proposition (Binet's equation)**.**

$$-mh^2u^2\left(\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2} + u\right) = F\left(\frac{1}{u}\right).$$

4.4 The Kepler problem

Proposition. The orbit of a planet around the sun is given by

$$r = \frac{\ell}{1 + e\cos\theta}, \tag{$*$}$$

with $\ell = h^2/k$ and $e = Ah^2/k$. This is the polar equation of a conic, with a focus (the sun) at the origin.

Law (Kepler's first law)**.** The orbit of each planet is an ellipse with the Sun at one focus.

Law (Kepler's second law)**.** The line between the planet and the sun sweeps out equal areas in equal times.

Law (Kepler's third law)**.** The square of the orbital period is proportional to the cube of the semi-major axis, or

$$P^2 \propto a^3.$$

4.5 Rutherford scattering

5 Rotating frames

5.1 Motion in rotating frames

Proposition. If S is an inertial frame, and S' is rotating relative to S with angular velocity $\boldsymbol{\omega}$, then

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)_S = \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)_{S'} + \boldsymbol{\omega} \times .$$

Applied to the position vector $\mathbf{r}(t)$ of a particle, it gives

$$\left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right)_S = \left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right)_{S'} + \boldsymbol{\omega} \times \mathbf{r}.$$

Proposition.

$$m\left(\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2}\right)_{S'} = \mathbf{F} - 2m\boldsymbol{\omega} \times \left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right)_{S'} - m\dot{\boldsymbol{\omega}} \times \mathbf{r} - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}).$$

5.2 The centrifugal force

5.3 The Coriolis force

6 Systems of particles

6.1 Motion of the center of mass

Proposition.

$$M\ddot{\mathbf{R}} = \mathbf{F}.$$

Proof.

$$\begin{aligned}
M\ddot{\mathbf{R}} &= \dot{\mathbf{P}} \\
&= \sum_{i=1}^{N} \dot{\mathbf{p}}_i \\
&= \sum_{i=1}^{N} \mathbf{F}_i^{\text{ext}} + \sum_{i=1}^{N}\sum_{j=1}^{N} \mathbf{F}_{ij} \\
&= \mathbf{F} + \frac{1}{2}\sum_i \sum_j (\mathbf{F}_{ij} + \mathbf{F}_{ji}) \\
&= \mathbf{F}
\end{aligned}$$

□

Law (Conservation of momentum)**.** If there is no external force, i.e. $\mathbf{F} = \mathbf{0}$, then $\dot{\mathbf{P}} = \mathbf{0}$. So the total momentum is conserved.

6.2 Motion relative to the center of mass

6.3 The two-body problem

6.4 Variable-mass problem

Proposition (Rocket equation)**.**

$$m\frac{\mathrm{d}v}{\mathrm{d}t} + u\frac{\mathrm{d}m}{\mathrm{d}t} = F.$$

7 Rigid bodies

7.1 Angular velocity

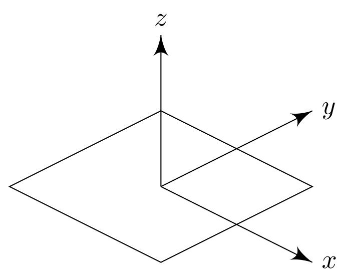
7.2 Moment of inertia

7.3 Calculating the moment of inertia

Theorem (Perpendicular axis theorem)**.** For a two-dimensional object (a lamina), and three perpendicular axes x, y, z through the same spot, with z normal to the plane,

$$I_z = I_x + I_y,$$

where I_z is the moment of inertia about the z axis.

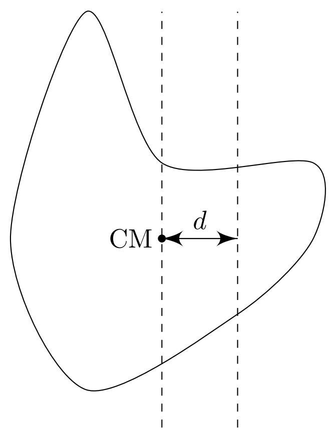


Proof. Let ρ be the mass per unit volume. Then

$$\begin{aligned} I_x &= \int \rho y^2 \;\mathrm{d}A \\ I_y &= \int \rho x^2 \;\mathrm{d}A \\ I_z &= \int \rho(x^2 + y^2) \;\mathrm{d}A = I_x + I_y. \end{aligned}$$ □

Theorem (Parallel axis theorem)**.** If a rigid body of mass M has moment of inertia I^C about an axis passing through the center of mass, then its moment of inertia about a parallel axis a distance d away is

$$I = I^C + Md^2.$$



Proof. With a convenient choice of Cartesian coordinates such that the center of mass is at the origin and the two rotation axes are $x = y = 0$ and $x = d, y = 0$,

$$I^C = \int \rho(x^2 + y^2)\,\mathrm{d}V,$$

and

$$\int \rho\mathbf{r}\,\mathrm{d}V = \mathbf{0}.$$

So

$$\begin{aligned}
I &= \int \rho((x - d)^2 + y^2)\,\mathrm{d}V \\
&= \int \rho(x^2 + y^2)\,\mathrm{d}V - 2d\int \rho x\,\mathrm{d}V + \int d^2\rho\,\mathrm{d}V \\
&= I^c + 0 + Md^2 \\
&= I^c + Md^2.
\end{aligned}$$

□

7.4 Motion of a rigid body

8 Special relativity

8.1 The Lorentz transformation

Law (Principle of Special Relativity)**.** Let S and S' be inertial frames, moving at the relative velocity of v. Then

$$x' = \gamma(x - vt)$$
$$t' = \gamma\left(t - \frac{v}{c^2}x\right),$$

where

$$\gamma = \frac{1}{\sqrt{1 - (v/c)^2}}.$$

This is the *Lorentz transformations* in the standard configuration (in one spatial dimension).

8.2 Spacetime diagrams

8.3 Relativistic physics

8.4 Geometry of spacetime

Proposition. All inertial observers agree on the value of Δs^2.

Proof.

$$\begin{aligned} c^2\Delta t'^2 - \Delta x'^2 &= c^2\gamma^2\left(\Delta t - \frac{v}{c^2}\Delta x\right)^2 - \gamma^2(\Delta x - v\Delta t)^2 \\ &= \gamma^2\left(1 - \frac{v^2}{c^2}\right)(c^2\Delta t - \Delta x^2) \\ &= c^2\Delta t - \Delta x^2. \end{aligned}$$

□

8.5 Relativistic kinematics

8.6 Particle physics