Example Sheet 1

1. In one spatial dimension, two frames of reference S and S' have coordinates (x,t) and (x',t') respectively. The coordinates are related by

$$x' = f(x,t) \qquad \text{and} \qquad t' = t\,.$$

Viewed in frame S, a particle follows a trajectory $x = x(t)$. It has velocity $v = \mathrm{d}x/\mathrm{d}t$ and acceleration $a = \mathrm{d}^2x/\mathrm{d}t^2$. Viewed in S', the trajectory is $x' = f(x(t),t)$. Using the chain rule, show that the velocity and acceleration of the particle in S' are given by

$$v' = \frac{\mathrm{d}x'}{\mathrm{d}t'} = v\frac{\partial f}{\partial x} + \frac{\partial f}{\partial t}\,, \qquad a' = \frac{\mathrm{d}^2x'}{\mathrm{d}t'^2} = a\frac{\partial f}{\partial x} + v^2\frac{\partial^2 f}{\partial x^2} + 2v\frac{\partial^2 f}{\partial x\,\partial t} + \frac{\partial^2 f}{\partial t^2}\,.$$

Suppose now that both S and S' are inertial frames. Explain why the function f must obey $\partial^2 f/\partial x^2 = \partial^2 f/\partial x\,\partial t = \partial^2 f/\partial t^2 = 0$. What is the most general form of f with these properties? Interpret this result.

2. A particle of mass m experiences a force field

$$\boldsymbol{F}(\boldsymbol{r}) = \left(-\frac{a}{r^2} + \frac{2b}{r^3}\right)\hat{\boldsymbol{r}}\,,$$

where $\hat{\boldsymbol{r}} = \boldsymbol{r}/r$ is a unit vector in the radial direction and a and b are positive constants. Show, by finding a potential energy $V(r)$ such that $\boldsymbol{F} = -\boldsymbol{\nabla}V$, that $\boldsymbol{F}$ is conservative. (You will need the result $\boldsymbol{\nabla}r = \hat{\boldsymbol{r}}$.)

Sketch $V(r)$ and describe qualitatively the possible motions of the particle moving in the radial direction, considering different initial positions and velocities. If the particle starts at $r = 2b/a$, what is the minimum speed that it must have in order to escape to infinity?

3. A satellite falls freely towards the Earth starting from rest at a distance R, much larger than the Earth's radius. Treating the Earth as a point of mass M, use dimensional analysis to show that the time T taken by the satellite to reach the Earth is given by

$$T = C\left(\frac{R^3}{GM}\right)^{1/2}\,,$$

where G is the gravitational constant and C is a dimensionless constant. (You will need the fact that the acceleration due to the Earth's gravitational field at a distance r from the centre of the Earth is GM/r^2.)

What is the conserved energy of the satellite? By integrating this equation, show that $C = \pi/2\sqrt{2}$.

4. A long time ago in a galaxy far, far away, a Death Star was constructed. Its surrounding force field caused any particle at position $\boldsymbol{r}$ relative to the centre of the Death Star to experience an acceleration

$$\ddot{\boldsymbol{r}} = \lambda\, \boldsymbol{r} \times \dot{\boldsymbol{r}}\,,$$

where λ is a constant. Show that the particle moves in this force field with constant speed. Show also that the magnitude of its acceleration is constant.

(a) A particle is projected *radially* with speed v from a point $\boldsymbol{r} = R\,\hat{\boldsymbol{r}}$ on the surface of the Death Star. Show that its trajectory is given by

$$\boldsymbol{r} = (R + vt)\,\hat{\boldsymbol{r}}\,.$$

(b) By considering the second derivative of $\boldsymbol{r} \cdot \boldsymbol{r}$ show that, for any particle moving in the force field, the distance r from the centre of the Death Star is given by

$$r^2 = v^2(t - t_0)^2 + r_0^2\,,$$

where t_0 and r_0 are constants and v is the speed of the particle. Obtain an expression for $\boldsymbol{r} \cdot \dot{\boldsymbol{r}}$ and show that $|\ddot{\boldsymbol{r}}| = \lambda r_0 v$.

5. A particle of mass m, charge q and position $\boldsymbol{x}(t)$ moves in both a uniform magnetic field $\boldsymbol{B}$, which points in a horizontal direction, and a uniform gravitational field $\boldsymbol{g}$, which points vertically downwards. Write down the equation of motion and show that it is invariant under translations $\boldsymbol{x} \mapsto \boldsymbol{x} + \boldsymbol{x}_0$. Show that

$$\dot{\boldsymbol{x}} = \omega\, \boldsymbol{x} \times \boldsymbol{n} + \boldsymbol{g}\, t + \boldsymbol{a}\,,$$

where $\omega = qB/m$ is the gyrofrequency, $\boldsymbol{n}$ is a unit vector in the direction of $\boldsymbol{B}$, and $\boldsymbol{a}$ is a constant vector. Show also that, with a suitable choice of origin, $\boldsymbol{a}$ can be written in the form $\boldsymbol{a} = a\,\boldsymbol{n}$.

By choosing suitable axes, show that the particle undergoes a helical motion together with a constant horizontal drift perpendicular to $\boldsymbol{B}$.

Suppose that you now wish to eliminate the drift by imposing a uniform electric field $\boldsymbol{E}$. Determine the direction and magnitude of $\boldsymbol{E}$.

6. At time $t = 0$, an insect of mass m jumps from a point O on the ground with velocity $\boldsymbol{v}$, while a wind blows with constant velocity $\boldsymbol{u}$. The gravitational acceleration is $\boldsymbol{g}$ and the air exerts a drag force on the insect equal to mk times the velocity of the wind *relative to the insect.*

(a) Show that the path of the insect is given by

$$\boldsymbol{x} = \left(\boldsymbol{u} + \frac{\boldsymbol{g}}{k}\right) t + \frac{1 - \mathrm{e}^{-kt}}{k} \left(\boldsymbol{v} - \boldsymbol{u} - \frac{\boldsymbol{g}}{k}\right)\,.$$

(b) In the case where the insect jumps vertically in a horizontal wind, show that the time T that elapses before it returns to the ground (which is also horizontal) satisfies

$$1 - \mathrm{e}^{-kT} = \frac{kT}{1+\lambda}\,,$$

where $\lambda = kv/g$. Find an expression for the horizontal range R in terms of λ, u and T. (Here $v = |\boldsymbol{v}|$, $g = |\boldsymbol{g}|$ and $u = |\boldsymbol{u}|$.)

7. A ball of mass m moves, under gravity, in a resistive medium that produces a frictional force of magnitude kv^2, where v is the ball's speed. If the ball is projected vertically upwards with initial speed u, show by dimensional analysis that when the ball returns to its point of projection, its speed w can be written in the form

$$w = uf(\lambda)\,,$$

where $\lambda = ku^2/mg$.

Integrate the equation of motion to show that $f(\lambda) = (1+\lambda)^{-1/2}$. [Hint: Thinking about v as a function of time may not be the easiest approach.]. Discuss what happens in the two extremes $\lambda \gg 1$ and $\lambda \ll 1$.

8*. The temperature $\theta(x,t)$ in a very long rod is governed by the one-dimensional diffusion equation

$$\frac{\partial\theta}{\partial t} = D\frac{\partial^2\theta}{\partial x^2}\,,$$

where D is a constant (the *thermal diffusivity* of the rod). At time $t=0$, the point $x=0$ is heated to a high temperature. At all later times, the conservation of heat energy implies that

$$Q = \int_{-\infty}^{\infty} \theta(x,t)\,\mathrm{d}x$$

is constant. Use dimensional analysis to show that $\theta(x,t)$ can be written in the form

$$\theta(x,t) = \frac{Q}{\sqrt{Dt}}\,F(z)\,,$$

where $z = x/\sqrt{Dt}$, and show further that

$$\frac{\mathrm{d}^2F}{\mathrm{d}z^2} + \frac{z}{2}\frac{\mathrm{d}F}{\mathrm{d}z} + \frac{1}{2}F = 0\,.$$

Integrate this equation once directly to obtain a first-order differential equation. Evaluate the constant of integration by considering either the symmetry of the problem or the behaviour of the solution as $z \to \pm\infty$. Hence show that, for $t > 0$,

$$\theta(x,t) = \frac{Q}{\sqrt{4\pi Dt}}\,\exp\left(-\frac{x^2}{4Dt}\right)\,.$$

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Example Sheet 2

1. A particle moves in a fixed plane and its position vector at time t is $\boldsymbol{r}$. Let (r, θ) be plane polar coordinates and let $\hat{\boldsymbol{r}}$ and $\hat{\boldsymbol{\theta}}$ be unit vectors in the directions of increasing r and increasing θ, respectively. Show that

$$\dot{\boldsymbol{r}} = \dot{r}\,\hat{\boldsymbol{r}} + r\dot{\theta}\,\hat{\boldsymbol{\theta}}\,.$$

The particle moves outwards with speed $v(t)$ on the equiangular spiral $r = a\,\exp(\theta\cot\alpha)$, where a and α are constants, with $0 < \alpha < \frac{1}{2}\pi$. Show that

$$v\sin\alpha = r\dot{\theta}\,,$$

and hence that

$$\dot{\boldsymbol{r}} = v\cos\alpha\,\hat{\boldsymbol{r}} + v\sin\alpha\,\hat{\boldsymbol{\theta}}\,.$$

Give an expression for $\ddot{\boldsymbol{r}}$ and show that $|\ddot{\boldsymbol{r}}|^2 = \dot{v}^2 + v^2\dot{\theta}^2$.

($*$) If $\dot{\theta}$ takes a constant value ω, show that the acceleration has magnitude v^2/r and is directed at an angle 2α to the position vector.

2. In these orbital questions, the particles move in a gravitational potential $\Phi_{\mathrm{g}}(r) = -k/r$ with $k > 0$.

In what follows, you should answer all questions using *only* energy and angular momentum conservation, and (for the circular orbits) the radial component of the equation of motion.

(a) Show that the radius, R, of the orbit of a satellite in geostationary orbit (in the equatorial plane) is approximately $(28)^{-2/3}R_{\mathrm{M}}$, where R_{M} is the radius of the Moon's orbit around the Earth.

(b) One particle moves in a parabolic orbit and another particle moves in a circular orbit. Show that if they pass through the same point then the ratio of their speeds at this point is $\sqrt{2}$. For a satellite orbiting the Earth in a circular orbit, what is the relationship between its orbital speed and its escape velocity?

If, instead of passing through the same point, the particles have the same angular momentum per unit mass, show that the periapsis distance of the parabola is half the radius of the circle.

(c) A particle moves with angular momentum h per unit mass in an ellipse, for which the distances from the focus to the periapsis (closest point to focus) and apoapsis (furthest point) are p and q, respectively. Show that

$$h^2\left(\frac{1}{p} + \frac{1}{q}\right) = 2k\,.$$

Show also that the speed V of the particle at the periapsis is related to the speed v of a particle moving in a circular orbit of radius p by $(1 + p/q)V^2 = 2v^2$.

(d) A particle P is initially at a very large distance from the origin moving with speed v on a trajectory that, in the absence of any force, would be a straight line for which the shortest distance from the origin is b. The shortest distance between P's actual trajectory and the origin is d. Show that $2kd = v^2(b^2 - d^2)$.

3. For a particle of mass m subject to an inverse-square force given by $\boldsymbol{F} = -mk\hat{\boldsymbol{r}}/r^2$, the vectors $\boldsymbol{h}$ and $\boldsymbol{e}$ are defined by

$$\boldsymbol{h} = \boldsymbol{r} \times \dot{\boldsymbol{r}}\,, \qquad \boldsymbol{e} = \frac{\dot{\boldsymbol{r}} \times \boldsymbol{h}}{k} - \frac{\boldsymbol{r}}{r}\,.$$

Show that $\boldsymbol{h}$ is constant and deduce that the particle moves in a plane through the origin.

The vector $\boldsymbol{e}$ is known as the *eccentricity vector* or *Laplace–Runge–Lenz vector*. Show that it too is constant and that

$$er\cos\theta = \frac{h^2}{k} - r\,,$$

where $e = |\boldsymbol{e}|$, $h = |\boldsymbol{h}|$ and θ is the angle between $\boldsymbol{r}$ and $\boldsymbol{e}$. Deduce that the orbit is a conic section.

4. A particle of unit mass moves with speed v in the gravitational field of the Sun and is influenced by radiation pressure. The forces acting on the particle are μ/r^2 towards the Sun and kv opposing the motion, where μ and k are constants. Write down the vector equation of motion and show that the vector $\boldsymbol{H}$, defined by

$$\boldsymbol{H} = \mathrm{e}^{kt}\,\boldsymbol{r} \times \dot{\boldsymbol{r}}\,,$$

is constant. Deduce that the particle moves in a plane through the origin. Establish the equations

$$r^2\dot{\theta} = h\,\mathrm{e}^{-kt} \qquad \text{and} \qquad \mu r = h^2\,\mathrm{e}^{-2kt} - r^3(\ddot{r} + k\dot{r})\,,$$

where r and θ are plane polar coordinates centred on the Sun and h is a constant. Show that, when $k = 0$, a circular orbit of radius a exists for any value of a, and find its angular frequency ω in terms of a and μ.

When $k/\omega \ll 1$, r varies so slowly that $\dot{r}$ and $\ddot{r}$ may be neglected in the above equations. Verify that in this case an approximate solution is

$$r = a\,\mathrm{e}^{-2kt}\,, \qquad \dot{\theta} = \omega\,\mathrm{e}^{3kt}\,.$$

Give a brief qualitative description of the behaviour of this solution for $t > 0$. Does the speed of the particle increase or decrease?

5. A particle P of unit mass moves in a plane under a central force

$$F(r) = -\frac{\lambda}{r^3} - \frac{\mu}{r^2}\,,$$

where λ and μ are positive constants. Write down the differential equation satisified by $u(\theta)$, where $u = 1/r$.

Given that P is projected with speed V from the point $r = r_0$, $\theta = 0$ in the direction perpendicular to OP, find the equation of the orbit under the assumptions

$$\lambda < V^2 r_0^2 < 2\mu r_0 + \lambda\,.$$

Explain the significance of these inequalities. Show that between consecutive apsides (points of greatest or least distance) the radius vector turns through an angle

$$\pi\left(1 - \frac{\lambda}{V^2 r_0^2}\right)^{-1/2}\,.$$

Under what condition is the orbit a closed curve?

6. A particle P of mass m moves under the influence of a central force of magnitude mk/r^3 directed towards a fixed point O. Initially $r = a$ and P has velocity v perpendicular to OP, where $v^2 < k/a^2$. Use the differential equation for the shape of the orbit to prove that P spirals in towards O (you should give the geometric equation of the spiral). Show also that it reaches O in a time

$$T = \frac{a^2}{\sqrt{k - a^2 v^2}}\,.$$

7*. A particle of mass m moves in a circular orbit of radius R under the influence of an attractive central force of magnitude $F(r)$. Obtain an equation relating R, $F(R)$, m and the orbital angular momentum per unit mass, h.

The particle experiences a very small radial perturbation of the form $u(\theta) = U + \epsilon(\theta)$, where $u = 1/r$ and $U = 1/R$. The orbital angular momentum is not affected. Obtain the equation for $\epsilon''(\theta)$. Given that the subsequent orbit is closed, show that

$$\frac{RF'(R)}{F(R)} = \beta^2 - 3\,,$$

where β is a rational number. Deduce that, if β is independent of R, then $F(r)$ is of the form Ar^α, where α is rational and greater than -3.

8. In these sequence of questions on the Coriolis force, use ω for the angular speed of the Earth, assume that events take place at latitude θ in the northern hemisphere and ignore centrifugal forces.

(a) Are bath-plug vortices in the northern hemisphere likely, on average, to be clockwise or anticlockwise?

(b) A straight river flows with speed v in a direction α degrees east of north. Show that the effect of the Coriolis force is to erode the right bank. Calculate the magnitude of the force.

(c) A plumb line is attached to the ceiling inside one of the carriages of a train and hangs down freely, at rest relative to the train. When the train is travelling at speed V in the north-easterly direction the plumb line hangs at an angle ϕ to the direction in which it hangs when the train is at rest. Ignoring centrifugal forces, show that $\phi \approx (2\omega V \sin\theta)/g$. Can the centrifugal force be ignored?

9. A bullet of mass m is fired from a point $\boldsymbol{r}_0$ with velocity $\boldsymbol{u}$ in a frame that rotates with constant angular velocity $\boldsymbol{\omega}$ relative to an inertial frame. The bullet is subject to a gravitational force $m\boldsymbol{g}$ which is constant in the rotating frame. Using the vector equation of motion and neglecting terms of order $|\boldsymbol{\omega}|^2$, show that the bullet's position vector measured in the rotating frame is approximately

$$\boldsymbol{r}_0 + \boldsymbol{u}\,t + \left(\tfrac{1}{2}\boldsymbol{g} - \boldsymbol{\omega}\times\boldsymbol{u}\right)t^2 + \tfrac{1}{3}\left(\boldsymbol{g}\times\boldsymbol{\omega}\right)t^3$$

at time t. Suppose that the bullet is projected from sea level on the Earth at latitude θ in the northern hemisphere, at an angle $\pi/4$ from the upward vertical and in a northward direction. Show that when the particle returns to sea level (neglecting the curvature of the Earth's surface), it has been deflected to the east by an amount approximately equal to

$$\frac{\sqrt{2}\,\omega|\boldsymbol{u}|^3}{3g^2}(3\sin\theta - \cos\theta)\,,$$

where ω is the angular speed of the Earth. Evaluate the approximate size of this deflection at latitude 52° N for $|\boldsymbol{u}| = 1000\,\mathrm{m\,s^{-1}}$.

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Example Sheet 3

1. A square hoop $ABCD$ is made of fine smooth wire and has side length $2a$. The hoop is horizontal and rotating with constant angular speed ω about a vertical axis through A. A small bead that can slide on the wire is initially at rest at the midpoint of the side BC. Choose axes fixed relative to the hoop, and let x be the distance of the bead from the vertex B on the side BC. Write down the position vector of the bead in the rotating frame.

Using the standard expression for acceleration in a rotating frame, show that

$$\ddot{x} - \omega^2 x = 0\,.$$

Hence show that the time that the bead takes to reach a corner of the hoop is $\omega^{-1}\cosh^{-1} 2$. Using dimensional analysis, explain why this time is independent of a.

Obtain an expression for the magnitude of the force exerted by the hoop on the bead.

2. In a system of particles, the ith particle has mass m_i and position vector $\boldsymbol{r}_i$ with respect to a fixed origin. The centre of mass of the system is at $\boldsymbol{R}$. Show that $\boldsymbol{L}$, the total angular momentum of the system about the origin, and $\boldsymbol{L}_{\rm c}$, the total angular momentum of the system about the centre of mass, are related by

$$\boldsymbol{L}_{\rm c} = \boldsymbol{L} - \boldsymbol{R} \times \boldsymbol{P}\,,$$

where $\boldsymbol{P}$ is the total linear momentum of the system.

Given that $\mathrm{d}\boldsymbol{P}/\mathrm{d}t = \boldsymbol{F}$, where $\boldsymbol{F}$ is the total external force, and $\mathrm{d}\boldsymbol{L}/\mathrm{d}t = \boldsymbol{G}$ where $\boldsymbol{G}$ is the total external torque about the origin, show that

$$\frac{\mathrm{d}\boldsymbol{L}_{\rm c}}{\mathrm{d}t} = \boldsymbol{G}_{\rm c}\,,$$

where $\boldsymbol{G}_{\rm c}$ is the total external torque about the centre of mass.

3. A system of particles with masses m_i and position vectors $\boldsymbol{r}_i$ $(i = 1, \ldots, n)$ moves under its own mutual gravitational attraction alone. Write down the equation of motion for $\boldsymbol{r}_i$. Show that a possible solution of the equations of motion is given by $\boldsymbol{r}_i = t^{2/3}\boldsymbol{a}_i$, where the vectors $\boldsymbol{a}_i$ are constant vectors satisfying

$$\boldsymbol{a}_i = \frac{9G}{2}\sum_{j\neq i}\frac{m_j(\boldsymbol{a}_i - \boldsymbol{a}_j)}{|\boldsymbol{a}_i - \boldsymbol{a}_j|^3}\,.$$

Show that, for this system, the total angular momentum about the origin and the total momentum both vanish. What is the angular momentum about any other fixed point?

4. Two particles of masses m_1 and m_2 move under their mutual gravitational attraction. Show from first principles that the quantity

$$\frac{1}{2}\dot{\boldsymbol{r}}\cdot\dot{\boldsymbol{r}} - \frac{GM}{r}$$

is constant, where $\boldsymbol{r}$ is the position vector of one particle relative to the other and $M = m_1 + m_2$.

The particles are released from rest a long way apart and fall towards each other. Show that the position of their centre of gravity is fixed, and that when they are a distance r apart their relative speed is $\sqrt{2GM/r}$.

($*$) When the particles are a distance a apart, they are given equal and opposite impulses (changes of momentum), each of magnitude I, and each perpendicular to the direction of motion. Show that subsequently $r^2\omega = aI/\mu$, where ω is the angular speed of either particle relative to the centre of mass and μ is the reduced mass of the system.

Show further that the minimum separation, d, of the two particles in the subsequent motion satisfies

$$(a^2 - d^2)I^2 = 2GM\mu^2 d\,.$$

5. A rocket, moving vertically upwards, ejects gas vertically downwards at speed u relative to the rocket. Derive the equation of motion

$$m\frac{\mathrm{d}v}{\mathrm{d}t} = -u\frac{\mathrm{d}m}{\mathrm{d}t} - mg\,,$$

where v and m are the speed and total mass of the rocket (including fuel) at time t. If u is constant and the rocket starts from rest with total mass m_0, show that

$$m = m_0\,\mathrm{e}^{-(gt+v)/u}\,.$$

6. A firework of initial mass m_0 is fired vertically upwards from the ground. Fuel is burnt at a constant rate $-\mathrm{d}m/\mathrm{d}t = \alpha$ and the exhaust is ejected at constant speed u relative to the firework. Show that the speed of the firework at time t, where $0 < t < m_0/\alpha$, is

$$v(t) = -gt - u\log\left(1 - \frac{\alpha t}{m_0}\right)\,,$$

and that this is positive provided $u > m_0 g/\alpha$.

Suppose now that nearly all of the firework consists of fuel, the mass of the containing shell being negligible. Show that the height attained by the shell when all of the fuel is burnt is

$$\frac{m_0}{\alpha}\left(u - \frac{m_0 g}{2\alpha}\right)\,.$$

7.

(a) Thin circular discs of radius a and b are made of uniform materials with mass per unit area ρ_a and ρ_b, respectively. They lie in the same plane. Their centres A and B are connected by a light rigid rod of length c. Find the moment of inertia of the system about an axis through B perpendicular to the plane of the discs.

(b) A thin uniform circular disc of radius a and centre A has a circular hole cut in it of radius b and centre B, where $AB = c < a - b$. The disc is free to oscillate in a vertical plane about a smooth fixed horizontal circular rod of radius b passing through the hole. Using the result of part (a), with ρ_b suitably chosen, show that the period of small oscillations is $2\pi\sqrt{l/g}$, where

$$l = c + \frac{a^4 - b^4}{2a^2 c}\,.$$

8. A yo-yo consists of two uniform discs, each of mass M and radius R, connected by a short light axle of radius a around which a portion of a thin string is wound. One end of the string is attached to the axle and the other to a fixed point P. The yo-yo is held with its centre of mass vertically below P and then released.

Assuming that the unwound part of the string remains approximately vertical, use the principle of conservation of energy to find the equation of motion of the centre of mass of the yo-yo. Find the tension in the string as the yo-yo falls.

If the string has length L, what is the speed of the yo-yo just before it reaches the end? Explain what happens next. What is the impulse due to the tension in the string at this time?

9. A uniform circular cylinder of mass M and radius a is free to turn about its axis which is horizontal. A thin uniform cylindrical shell of mass $M/2$ and radius a is fitted over the cylinder. At time $t = 0$ the angular velocity of the cylinder is Ω, while the shell is at rest. The shell exerts a frictional torque on the cylinder of magnitude $k(\omega - \varpi)$, where $\omega(t)$ and $\varpi(t)$ are the angular velocities of the cylinder and shell, respectively, at time t about the axis. Prove that

$$\omega(t) = \frac{1}{2}\Omega\left(1 + \mathrm{e}^{-4kt/Ma^2}\right)\,,$$

and find the corresponding expression for $\varpi(t)$.

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Example Sheet 4

1. A clock C is at rest at the spatial origin of an inertial frame S. A second clock C' is at rest at the spatial origin of an inertial frame S' moving with constant speed u relative to S. The clocks read $t = t' = 0$ when the two spatial origins coincide.

When C' reads t'_2 it receives a radio signal from C sent when C reads t_1. Draw a space-time diagram showing this process.

Determine the space-time coordinates (ct_2, x_2) in S of the point (event) at which C' receives the radio signal. Hence show that

$$t_1 = t'_2 \sqrt{\frac{1 - u/c}{1 + u/c}}\,.$$

2. S and S' are inertial frames in a two-dimensional space-time, the origins of which coincide. Observers O and O' are at the spatial origins of S and S' respectively. Observer O' moves at velocity u relative to observer O, where $u > 0$. Observer O observes a particle P passing through the origin and moving with velocity v, where $v < u$. Observer O' observes P moving with velocity $-v$.

Draw a space-time diagram, from O's point of view, to illustrate this situation. Use the relativistic velocity transformation law to show that

$$v = \frac{c^2}{u}\left(1 - \sqrt{1 - \frac{u^2}{c^2}}\right).$$

Writing $u/c = \tanh\varphi$, find an expression for v/c in terms of the rapidity φ.

3. Write down the Lorentz transformation law for the energy and relativistic 3-momentum of a particle.

In an inertial frame S, a photon with energy E moves in the xy plane at an angle θ with respect to the x axis. Show that in a second frame S' whose relative speed is u directed in the x direction, the energy and angle are given by

$$E' = \gamma E(1 - \beta\cos\theta) \qquad \text{and} \qquad \cos\theta' = \frac{\cos\theta - \beta}{1 - \beta\cos\theta}\,,$$

where $\beta = u/c$ and $\gamma = (1 - \beta^2)^{-1/2}$. Write down E and $\cos\theta$ as functions of E' and $\cos\theta'$.

For a photon moving in the x direction, derive the *relativistic Doppler effect* by showing that there is a frequency change by a factor

$$\sqrt{\frac{1-\beta}{1+\beta}}\,.$$

($*$) How does this relate to question 1?

A source of photons is at rest in S'. Derive the *headlight effect* by showing that, as $u \to c$, the photons emitted in the forward direction ($\cos\theta' > 0$) in S are concentrated in a narrow cone about $\theta = 0$.

($*$) Show that the semi-angle of the cone is approximately $\sqrt{2(1-\beta)}$.

4. Pulsars are stars that emit pulses of radiation at regular intervals. Bob and Ann count pulses from a very distant pulsar in the y direction. Ann travels at a speed given by $\beta \equiv u/c = 24/25$ in the x direction for seven years and then comes back at the same speed, while Bob stays at home. At the end of the trip they have counted the same number of pulses. Use question 3 (with $\theta = \frac{1}{2}\pi$) to show that on return she has aged by 14 years and he by 50.

Obtain the corresponding result when the pulsar is in the x direction (question 3 with $\theta = 0$), drawing a space-time diagram to show why Bob and Ann count the same number of pulses.

5. A particle with 4-momentum P is detected by an observer whose four-velocity is U. Working in the rest frame of the observer, express $P\cdot U$ in terms of the mass of the particle and its speed v in the observer's rest frame. Show that

$$\frac{v}{c} = \sqrt{1 - \frac{(P\cdot P)c^2}{(P\cdot U)^2}}\,.$$

6. A particle of rest mass m_0 disintegrates into two particles of rest masses m_1 and m_2. Use conservation of relativistic energy and relativistic 3-momentum to show that the energies E_1 and E_2 of the particles in the rest frame of the original particle are given by

$$\frac{E_1}{c^2} = \frac{m_0^2 + m_1^2 - m_2^2}{2m_0} \qquad \text{and} \qquad \frac{E_2}{c^2} = \frac{m_0^2 + m_2^2 - m_1^2}{2m_0}\,.$$

Let P_0 be the 4-momentum of the original particle, and let and P_1 and P_2 be the 4-momenta of the product particles. Derive the above results by expanding the invariant quantities $(P_0 - P_1)\cdot(P_0 - P_1)$ and $(P_0 - P_2)\cdot(P_0 - P_2)$.

7. A photon (of zero rest mass) collides with an electron of rest mass m that is initially at rest in the laboratory frame. Show that the angle θ by which the photon is deflected (measured in the laboratory frame) is related to the magnitudes p and q of its initial and final momenta by

$$2\sin^2(\tfrac{1}{2}\theta) = \frac{mc}{q} - \frac{mc}{p}\,.$$

8. In a laboratory frame a particle of rest mass m_1 has energy E_1, and a second particle of rest mass m_2 is at rest. By considering the scalar quantity $(P_1 + P_2)\cdot(P_1 + P_2)$, or otherwise, show that the combined energy in the centre-of-momentum frame (i.e. the frame in which the total 3-momentum is zero) is

$$\sqrt{m_1^2c^4 + m_2^2c^4 + 2E_1m_2c^2}\,.$$

Hence show that, in a collision between a proton with energy E and a proton at rest, it is not possible to create a proton-antiproton pair (in addition to the original protons) if $E < 7mc^2$, where m is the rest mass of the proton and also of the antiproton.

9. A rocket ejects exhaust at constant speed u relative to itself by a process that conserves relativistic energy and momentum (but not mass!). Let m be the rest mass of the rocket when the rocket has speed v measured in the *initial* rest frame of the rocket.

By equating the 4-momentum P of the rocket with the total 4-momentum at a later time when the 4-momentum of the rocket is $P + \delta P$, show that

$$\frac{\mathrm{d}(m\gamma v)}{\mathrm{d}v} = \left(\frac{v-u}{1-uv/c^2}\right)\frac{\mathrm{d}(m\gamma)}{\mathrm{d}v}\,,$$

where $\gamma = (1 - v^2/c^2)^{-1/2}$. Rearrange this as a differential equation for $m\gamma$ and derive the relativistic rocket equation

$$v = c\tanh\left(\frac{u}{c}\log r\right),$$

where $r = m_0/m$ and m_0 is the initial rest mass of the rocket.

Now suppose that the rocket has an ideal photon drive, so that all matter to be ejected is converted first to photons by means of positron-electron annihilation. By considering directly the initial and final energy and momentum (taking into account the total energy and momentum of the ejected photons, which you do not need to calculate), derive the relativistic rocket equation for the case of a photon-drive rocket.

Please send any comments and corrections to `gio10@cam.ac.uk`