Part IA — Dynamics and Relativity
Definitions

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These notes are not endorsed by the lecturers, and I have modified them (often significantly) after lectures. They are nowhere near accurate representations of what was actually lectured, and in particular, all errors are almost surely mine.

Familiarity with the topics covered in the non-examinable Mechanics course is assumed.

Basic concepts
Space and time, frames of reference, Galilean transformations. Newton's laws. Dimensional analysis. Examples of forces, including gravity, friction and Lorentz. [4]

Newtonian dynamics of a single particle
Equation of motion in Cartesian and plane polar coordinates. Work, conservative forces and potential energy, motion and the shape of the potential energy function; stable equilibria and small oscillations; effect of damping.

Angular velocity, angular momentum, torque.

Orbits: the $u(\theta)$ equation; escape velocity; Kepler's laws; stability of orbits; motion in a repulsive potential (Rutherford scattering). Rotating frames: centrifugal and coriolis forces. *Brief discussion of Foucault pendulum.* [8]

Newtonian dynamics of systems of particles
Momentum, angular momentum, energy. Motion relative to the centre of mass; the two body problem. Variable mass problems; the rocket equation. [2]

Rigid bodies
Moments of inertia, angular momentum and energy of a rigid body. Parallel axes theorem. Simple examples of motion involving both rotation and translation (e.g. rolling). [3]

Special relativity
The principle of relativity. Relativity and simultaneity. The invariant interval. Lorentz transformations in $(1+1)$-dimensional spacetime. Time dilation and length contraction. The Minkowski metric for $(1+1)$-dimensional spacetime. Lorentz transformations in $(3+1)$ dimensions. 4-vectors and Lorentz invariants. Proper time. 4-velocity and 4-momentum. Conservation of 4-momentum in particle decay. Collisions. The Newtonian limit. [7]

Contents

0 Introduction

1 Newtonian dynamics of particles

Definition (Particle)**.** A *particle* is an object of insignificant size, hence it can be regarded as a point. It has a *mass* $m > 0$, and an *electric charge* q.

Its position at time t is described by its *position vector*, $\mathbf{r}(t)$ or $\mathbf{x}(t)$ with respect to an origin O.

Definition (Frame of reference)**.** A *frame of reference* is a choice of coordinate axes for $\mathbf{r}$.

Definition (Velocity)**.** The *velocity* of the particle is

$$\mathbf{v} = \dot{\mathbf{r}} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}.$$

Definition (Acceleration)**.** The *acceleration* of the particle is

$$\mathbf{a} = \dot{\mathbf{v}} = \ddot{\mathbf{r}} = \frac{\mathrm{d}^2\mathbf{r}}{dt^2}.$$

Definition (Momentum)**.** The *momentum* of a particle is

$$\mathbf{p} = m\mathbf{v} = m\dot{\mathbf{r}}.$$

m is the *inertial mass* of the particle, and measures its reluctance to accelerate, as described by Newton's second law.

1.1 Newton's laws of motion

Definition (Inertial frames)**.** *Inertial frames* are frames of references in which the frames themselves are not accelerating. Newton's Laws only hold in inertial frames.

1.2 Galilean transformations

Definition (Galilean boost)**.** A *Galilean boost* is a change in frame of reference by

$$\begin{aligned}\mathbf{r}' &= \mathbf{r} - \mathbf{v}t\\ t' &= t\end{aligned}$$

for a fixed, constant $\mathbf{v}$.

1.3 Newton's Second Law

2 Dimensional analysis

2.1 Units

2.2 Scaling

3 Forces

3.1 Force and potential energy in one dimension

Definition (Potential energy)**.** Given a force field $F = F(x)$, we define the *potential energy* to be a function $V(x)$ such that

$$F = -\frac{\mathrm{d}V}{\mathrm{d}x}.$$

or

$$V = -\int F \,\mathrm{d}x.$$

V is defined up to a constant term. We usually pick a constant of integration such that the potential drops to 0 at infinity.

Definition (Total energy)**.** The *total energy* of a system is given by

$$E = T + V,$$

where V is the potential energy and $T = \frac{1}{2}m\dot{x}^2$ is the kinetic energy.

3.2 Motion in a potential

Definition (Equilibrium point)**.** A particle is in *equilibrium* if it has no tendency to move away. It will stay there for all time. Since $m\ddot{x} = -V'(x)$, the equilibrium points are the stationary points of the potential energy, i.e.

$$V'(x_0) = 0.$$

Definition (Kinetic energy)**.** We define the *kinetic energy* of the particle to be

$$T = \frac{1}{2}m|\mathbf{v}|^2 = \frac{1}{2}m\dot{\mathbf{r}}\cdot\dot{\mathbf{r}}.$$

Definition (Power)**.** The *power* is the rate at which work is done on a particle by a force. It is given by

$$P = \mathbf{F}\cdot\mathbf{v}.$$

Definition (Work done)**.** The *work done* on a particle by a force is the change in kinetic energy caused by the force. The work done on a particle moving from $\mathbf{r}_1 = \mathbf{r}(t_1)$ to $\mathbf{r}_2 = \mathbf{r}(t_2)$ along a trajectory C is the line integral

$$W = \int_C \mathbf{F}\cdot\mathrm{d}\mathbf{r} = \int_{t_1}^{t_2} \mathbf{F}\cdot\dot{\mathbf{r}}\,\mathrm{d}t = \int_{t_1}^{t_2} P\,\mathrm{d}t.$$

Definition (Conservative force and potential energy)**.** A *conservative force* is a force field $\mathbf{F}(\mathbf{r})$ that can be written in the form

$$\mathbf{F} = -\nabla V.$$

V is the *potential energy function.*

3.3 Central forces

Definition (Central force)**.** A *central force* is a force with a potential $V(r)$ that depends only on the distance from the origin, $r = |\mathbf{r}|$. Note that a central force can be either attractive or repulsive.

Definition (Angular momentum)**.** The *angular momentum* of a particle is

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} = m\mathbf{r} \times \dot{\mathbf{r}}.$$

Definition (Torque)**.** The *torque* G of a particle is the rate of change of angular momentum.

$$\mathbf{G} = \frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t} = \mathbf{r} \times \mathbf{F}.$$

3.4 Gravity

Definition (Gravitaional potential and field)**.** The *gravitational potential* is the gravitational potential energy per unit mass. It is

$$\Phi_g(r) = -\frac{GM}{r}.$$

Note that *potential* is confusingly different from *potential energy.*

If we have a second particle, the potential *energy* is given by $V = m\Phi_g$.

The *gravitational field* is the force per unit mass,

$$\mathbf{g} = -\nabla\Phi_g = -\frac{GM}{r^2}\hat{\mathbf{r}}.$$

3.5 Electromagnetism

Definition (Electrostatic potential)**.** The electrostatic potential is a function $\Phi_e(\mathbf{r})$ such that

$$\mathbf{E} = -\nabla\Phi_e.$$

Definition (Electric constant)**.** ε_0 is the *electric constant* or *vacuum permittivity* or *permittivity of free space.*

3.6 Friction

4 Orbits

4.1 Polar coordinates in the plane

Definition (Radial and angular velocity)**.** $\dot{r}$ is the *radial velocity*, and $\dot{\theta}$ is the *angular velocity*.

4.2 Motion in a central force field

Notation (Angular momentum per unit mass)**.** The *angular momentum per unit mass* is

$$h = \frac{L}{m} = r^2\dot{\theta} = \text{const}.$$

Definition (Periapsis, apoapsis and apsides)**.** The points of minimum and maximum r in such an orbit are called the *periapsis* and *apoapsis*. They are collectively known as the *apsides*.

Definition (Perihelion and aphelion)**.** For an orbit around the Sun, the periapsis and apoapsis are known as the *perihelion* and *aphelion*.

Definition (Perigee and apogee)**.** The perihelion and aphelion of the Earth are known as the *perigee* and *apogee*.

4.3 Equation of the shape of the orbit

Notation.

$$u = \frac{1}{r}.$$

4.4 The Kepler problem

Definition (Eccentricity)**.** The dimensionless parameter $e \geq 0$ in the equation of orbit is the *eccentricity* and determines the shape of the orbit.

4.5 Rutherford scattering

5 Rotating frames

5.1 Motion in rotating frames

Definition (Angular velocity vector)**.** The *angular velocity vector* of a rotating frame is $\boldsymbol{\omega} = \omega \hat{\mathbf{z}}$, where $\hat{\mathbf{z}}$ is the axis of rotation and ω is the angular speed.

Definition (Fictious forces)**.** The additional terms on the RHS of the equation of motion in rotating frames are *fictitious forces*, and are needed to explain the motion observed in S'. They are

- *Coriolis force*: $-2m\boldsymbol{\omega} \times \left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right)_{S'}$.
- *Euler force*: $-m\dot{\boldsymbol{\omega}} \times \mathbf{r}$
- *Centrifugal force*: $-m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$.

5.2 The centrifugal force

5.3 The Coriolis force

6 Systems of particles

6.1 Motion of the center of mass

Definition (Total mass)**.** The *total mass* of the system is $M = \sum m_i$.

Definition (Center of mass)**.** The *center of mass* is located at

$$\mathbf{R} = \frac{1}{M}\sum_{i=1}^{N} m_i \mathbf{r}_i.$$

This is the mass-weighted average position.

Definition (Total linear momentum)**.** The *total linear momentum* is

$$\mathbf{P} = \sum_{i=1}^{N} \mathbf{p}_i = \sum_{i=1}^{N} m_i \dot{\mathbf{r}}_i = M\dot{\mathbf{R}}.$$

Note that this is equivalent to the momentum of a single particle of mass M at the center of mass.

Definition (Total external force)**.** The *total external force* is

$$\mathbf{F} = \sum_{i=1}^{N} \mathbf{F}_i^{\text{ext}}.$$

Definition (Center of mass frame)**.** The *center of mass frame* is an inertial frame in which $\mathbf{R} = 0$ for all time.

Definition (Total angular momentum)**.** The *total angular momentum* of the system about the origin is

$$\mathbf{L} = \sum_i \mathbf{r}_i \times \mathbf{p}_i.$$

Definition (Total external torque)**.** The *total external torque* is

$$\mathbf{G} = \sum_i \mathbf{r}_i \times \mathbf{F}_i^{\text{ext}}.$$

6.2 Motion relative to the center of mass

6.3 The two-body problem

6.4 Variable-mass problem

7 Rigid bodies

Definition (Rigid body)**.** A *rigid body* is an extended object, consisting of N particles that are constrained such that the distance between any pair of particles, $|\mathbf{r}_i - \mathbf{r}_j|$, is fixed.

7.1 Angular velocity

Definition (Moment of inertia)**.** The *moment of inertia* of a particle is

$$I = ms^2 = m|\hat{\mathbf{n}} \times \mathbf{r}|^2,$$

where s is the distance of the particle from the axis of rotation.

7.2 Moment of inertia

Definition (Moment of inertia)**.** The *moment of inertia* of a rigid body about the rotation axis $\hat{\mathbf{n}}$ is

$$I = \sum_{i=1}^{N} m_i s_i^2 = \sum_{i=1}^{N} m_i |\hat{\mathbf{n}} \times \mathbf{r}_i|^2.$$

Definition. The *angular momentum* is

$$\mathbf{L} = \sum_i m_i \mathbf{r}_i \times \dot{\mathbf{r}}_i = \sum_i m_i \mathbf{r}_i \times (\boldsymbol{\omega} \times \mathbf{r}_i).$$

7.3 Calculating the moment of inertia

Definition (Mass, center of mass and moment of inertia)**.** The *mass* is

$$M = \int \rho \;\mathrm{d}V.$$

The *center of mass* is

$$\mathbf{R} = \frac{1}{M} \int \rho \mathbf{r} \;\mathrm{d}V$$

The *moment of inertia* is

$$I = \int \rho s^2 \;\mathrm{d}V = \int \rho |\hat{\mathbf{n}} \times \mathbf{r}|^2 \;\mathrm{d}V.$$

7.4 Motion of a rigid body

8 Special relativity

8.1 The Lorentz transformation

Definition (Lorentz factor)**.** The *Lorentz factor* is

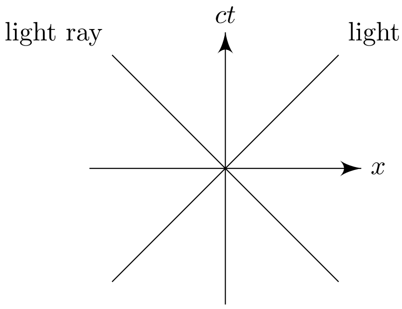
$$\gamma = \frac{1}{\sqrt{1-(v/c)^2}}.$$

8.2 Spacetime diagrams

Definition (Spacetime)**.** The union of space and time in special relativity is called *Minkowski spacetime.* Each point P represents an *event*, labelled by coordinates (ct, x) (note the order!).

A particle traces out a *world line* in spacetime, which is straight if the particle moves uniformly.

Light rays moving in the x direction have world lines inclined at 45°.



8.3 Relativistic physics

Definition (Simultaneous events)**.** We say two events P_1 and P_2 are simultaneous in the frame S if $t_1 = t_2$.

Definition (Proper length)**.** The *proper length* is the length measured in an object's rest frame.

8.4 Geometry of spacetime

Definition (Invariant interval)**.** The *invariant interval* or *spacetime interval* between P and Q is defined as

$$\Delta s^2 = c^2\Delta t^2 - \Delta x^2.$$

Note that this quantity Δs^2 can be both positive or negative — so Δs might be imaginary!

Definition (Line element)**.** The *line element* is

$$\mathrm{d}s^2 = c^2\mathrm{d}t^2 - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2.$$

Definition (Timelike, spacelike and lightlike separation)**.** Events with $\Delta s^2 > 0$ are *timelike separated.* It is possible to find inertial frames in which the two events occur in the same position, and are purely separated by time. Timelike-separated events lie within each other's light cones and can influence one another.

Events with $\Delta s^2 < 0$ are *spacelike separated.* It is possible to find inertial frame in which the two events occur in the same time, and are purely separated by space. Spacelike-separated events lie out of each other's light cones and cannot influence one another.

Events with $\Delta s^2 = 0$ are *lightlike* or *null separated.* In all inertial frames, the events lie on the boundary of each other's light cones. e.g. different points in the trajectory of a photon are lightlike separated, hence the name.

Definition (Rapidity)**.** The *rapidity* of a Lorentz boot is ϕ such that

$$\beta = \tanh\phi, \quad \gamma = \cosh\phi, \quad \gamma\beta = \sinh\phi.$$

8.5 Relativistic kinematics

Definition (Proper time)**.** The *proper time* τ is defined such that

$$\Delta\tau = \frac{\Delta s}{c}$$

τ is the time experienced by the particle, i.e. the time in the particles rest frame.

Definition (Position 4-vector and 4-velocty)**.** The *position 4-vector* is

$$X(\tau) = \begin{pmatrix} ct(\tau) \\ \mathbf{x}(\tau) \end{pmatrix}.$$

Its *4-velocity* is defined as

$$U = \frac{\mathrm{d}X}{\mathrm{d}\tau} = \begin{pmatrix} c\frac{\mathrm{d}t}{\mathrm{d}\tau} \\ \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}\tau} \end{pmatrix} = \frac{\mathrm{d}t}{\mathrm{d}\tau}\begin{pmatrix} c \\ \mathbf{u} \end{pmatrix} = \gamma_u \begin{pmatrix} c \\ \mathbf{u} \end{pmatrix},$$

where $\mathbf{u} = \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}$.

Definition (4-vector)**.** A *4-vector* is a 4-component vectors that transforms in this way under a Lorentz transformation, i.e. $X' = \Lambda X$.

When using suffix notation, the indices are written above (superscript) instead of below (subscript). The indices are written with Greek letters which range from 0 to 3. So we have X^μ instead of X_i, for $\mu = 0, 1, 2, 3$. If we write X_μ instead, it means a different thing. This will be explained more in-depth in the electromagnetism course (and you'll get more confused!).

Definition (4-momentum)**.** The *4-momentum* of a particle of mass m is

$$P = mU = m\gamma_u \begin{pmatrix} c \\ \mathbf{u} \end{pmatrix}$$

The 4-momentum of a system of particles is the sum of the 4-momentum of the particles, and is conserved in the absence of external forces.

The spatial components of P are the *relativistic 3-momentum*,

$$\mathbf{p} = m\gamma_u\mathbf{u},$$

which differs from the Newtonian expression by a factor of γ_u. Note that $|\mathbf{p}| \to \infty$ as $|\mathbf{u}| \to c$.

Definition (Relativistic energy)**.** The *relativistic energy* of a particle is $E = P^0 c$. So

$$E = m\gamma c^2 = mc^2 + \frac{1}{2}m|\mathbf{u}|^2 + \cdots$$

Definition (4-force)**.** The *4-force* is

$$F = \frac{\mathrm{d}P}{\mathrm{d}\tau}$$

8.6 Particle physics

Definition (Center of momentum frame)**.** The *center of momentum (CM) frame*, or *zero momentum frame*, is an inertial frame in which the total 3-momentum is $\sum \mathbf{p} = 0$.