ANALYSIS 1 EXAMPLES SHEET 1

Lent Term 2015 W. T. G.

1. Let (a_n) and (b_n) be two real sequences. Suppose that (a_n) is a subsequence of (b_n) and (b_n) is a subsequence of (a_n). Does it follow that they are the same sequence?

2. For each positive integer k let $a_{2^k} = 1$ and for every n that is not a power of 2, let $a_n = 0$. Prove directly from the definition of convergence that the sequence (a_n) does not converge.

3. Let (a_n) be a real sequence. We say that $a_n \to \infty$ if for every K there exists N such that for every $n \geq N$ we have $a_n \geq K$.

(i) Write down a similar definition for $a_n \to -\infty$.

(ii) Show that $a_n \to -\infty$ if and only if $-a_n \to \infty$.

(iii) Suppose that no a_n is 0. Prove that if $a_n \to \infty$, then $\frac{1}{a_n} \to 0$.

(iv) Again suppose that no a_n is 0. If $\frac{1}{a_n} \to 0$, does it follow that $a_n \to \infty$?

4. Let $a_1 > b_1 > 0$ and for every $n \geq 1$ let $a_{n+1} = (a_n + b_n)/2$ and let $b_{n+1} = 2a_n b_n/(a_n + b_n)$. Show that $a_n > a_{n+1} > b_{n+1} > b_n$. Deduce that the two sequences converge to a common limit. What is that limit?

5. Let $(a_1, b_1) \supset (a_2, b_2) \supset \dots$ be a nested sequence of non-empty open intervals. Must $\bigcap_{n=1}^{\infty}(a_n, b_n)$ be non-empty? If not, then find a (non-trivial) additional condition that guarantees that the intersection is non-empty.

6. (i) Let (a_n) be a real sequence that is bounded but that does not converge. Prove that it has two convergent subsequences with different limits.

(ii) Prove that every real sequence has a subsequence that converges or tends to $\pm\infty$.

7. Let a be a real number and let (a_n) be a sequence such that every subsequence of (a_n) has a further subsequence that converges to a. Prove that $a_n \to a$.

8. Let (a_n) be a Cauchy sequence. Prove that (a_n) has a subsequence (a_{n_k}) such that $|a_{n_p} - a_{n_q}| < 2^{-p}$ whenever $p \leq q$.

9. Let $f : \mathbb{R} \to (0, \infty)$ be a decreasing function. (That is, if $x < y$ then $f(x) \geq f(y)$.) Define a sequence (a_n) inductively by $a_1 = 1$ and $a_{n+1} = a_n + f(a_n)$ for every $n \geq 1$. Prove that $a_n \to \infty$.

10. Investigate the convergence of the following series. For each expression that contains the variable z, find all complex numbers z for which the series converges.

$$\sum_n \frac{\sin n}{n^2} \qquad \sum_n \frac{n^2 z^n}{5^n} \qquad \sum_n \frac{(-1)^n}{4+\sqrt{n}} \qquad \sum_n \frac{z^n(1-z)}{n} \qquad \sum_{n\geq 3} \frac{n^2}{(\log\log n)^{\log n}}$$

11. The two series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots$ and $1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \dots$ have the same terms but in different orders. Let S_n and T_n be the partial sums to n terms. Let $H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$. Show that $S_{2n} = H_{2n} - H_n$ and $T_{3n} = H_{4n} - \frac{1}{2}H_{2n} - \frac{1}{2}H_n$. Show that the sequence (S_n) converges to a limit S and that $T_n \to 3S/2$.

12. Prove that $\sum_n \frac{1}{n(\log n)^\alpha}$ converges if $\alpha > 1$ and diverges otherwise. Does the series $\sum_n \frac{1}{n \log n \log\log n}$ converge?

13. Let (a_n) be a sequence of positive real numbers such that $\sum_n a_n$ diverges. Prove that there exists a sequence (b_n) of positive real numbers such that $b_n/a_n \to 0$, but $\sum_n b_n$ is still divergent.

14. Let x be a real number and let $\sum_n a_n$ be a series that converges but that does not converge absolutely. Prove that the terms can be reordered so that the series converges to x. That is, show that there is a bijection $\pi : \mathbb{N} \to \mathbb{N}$ such that $\sum_n a_{\pi(n)} = x$.

15. For every positive integer k write $\log_k(x)$ for $\log\log\dots\log(x)$, where the logarithm has been taken k times. (Thus, $\log_1(x) = \log x$, $\log_2(x) = \log\log x$, and so on.) Define a function $f : \mathbb{N} \to \mathbb{R}$ by taking $f(n)$ to be $n \log n \log_2 n \dots \log_{k(n)} n$, where $k(n)$ is the largest integer such that $\log_{k(n)} n \geq 1$. Does the series $\sum_n \frac{1}{f(n)}$ converge?

16. Can the open interval $(0, 1)$ be written as a union of disjoint closed intervals of positive length?

Any comments or queries can be sent to wtg10@dpmms.cam.ac.uk.

ANALYSIS 1 EXAMPLES SHEET 2

Lent Term 2015 W. T. G.

1. Let (a_n) and (b_n) be two real sequences. Suppose that (a_n) is a subsequence of (b_n) and (b_n) is a subsequence of (a_n). Suppose also that (a_n) converges. Does it follow that they are the same sequence?

2. Let $H : \mathbb{R} \to \mathbb{R}$ be defined as follows: if $x < 0$ then $H(x) = 0$ and if $x \geq 0$ then $H(x) = 1$. Prove carefully that H is not continuous (i) by directly using the definition of continuity and (ii) by using the sequence definition.

3. Suppose that $f(x) \to \ell$ as $x \to a$ and $g(y) \to k$ as $y \to \ell$. Does it follow that $g(f(x)) \to k$ as $x \to a$?

4. For each natural number n, let $f_n : [0, 1] \to [0, 1]$ be a continuous function, and for each n let h_n be defined by $h_n(x) = \max\{f_1(x), \dots, f_n(x)\}$. Show that for each n the function h_n is continuous on $[0, 1]$. Must the function h defined by $h(x) = \sup\{f_n(x) : n \in \mathbb{N}\}$ be continuous?

5. Let $g : [0, 1] \to [0, 1]$ be a continuous function. Prove that there exists some $c \in [0, 1]$ such that $g(c) = c$. Such a c is called a *fixed point* of g.

Give an example of a bijection $h : [0, 1] \to [0, 1]$ with no fixed point.

Give an example of a continuous bijection $p : (0, 1) \to (0, 1)$ with no fixed point.

6. Prove that the function $q(x) = 2x^5 + 3x^4 + 2x + 16$ (defined on the reals) takes the value 0 exactly once, and that the number where it takes that value is somewhere in the interval $[-2, -1]$.

7. Prove rigorously that there are exactly nine solutions to the simultaneous equations $x = 1000(y^3 - y)$ and $y = 1000(x^3 - x)$. That is, prove that there are exactly nine ordered pairs (x, y) such that the two equations are satisfied.

8. Let $f : [0, 1] \to \mathbb{R}$ be continuous, with $f(0) = f(1) = 0$. Suppose that for every $x \in (0, 1)$ there exists $\delta > 0$ such that both $x + \delta$ and $x - \delta$ belong to $(0, 1)$ and $f(x) = \frac{1}{2}(f(x - \delta) + f(x + \delta))$. Prove that $f(x) = 0$ for every $x \in [0, 1]$.

9. Define a function $f : \mathbb{R} \to \mathbb{R}$ as follows. If x is irrational, then $f(x) = 0$, while if x is rational, then $f(x) = 1/q$, where q is the denominator of x. (That is, $x = p/q$, with p and q coprime integers and $q > 0$.) Prove that f is continuous at every irrational and discontinuous at every rational.

10. Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Which of the following statements are always true and which are sometimes false?
(i) If f is increasing, then $f'(x) \geq 0$ for every $x \in (a, b)$.
(ii) If $f'(x) \geq 0$ for every $x \in (a, b)$, then f is increasing.
(iii) If f is strictly increasing, then $f'(x) > 0$ for every $x \in (a, b)$.
(iv) If $f'(x) > 0$ for every $x \in (a, b)$, then f is strictly increasing.

11. (i) Let $g : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that $g(0) = g'(0) = 0$ and $g''(0)$ exists and is positive. Prove that there exists $x > 0$ such that $g(x) > 0$.
(ii) Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that $f(0) = 0$, and $f''(0)$ exists and is positive. Prove that there exists $x > 0$ such that $f(2x) > 2f(x)$.

12. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable everywhere. Prove that if $f'(x) \to \ell$ as $x \to \infty$, then $f(x)/x \to \ell$. If $f(x)/x \to \ell$ as $x \to \infty$, does it follow that $f'(x) \to \ell$?

13. Find a function $f : \mathbb{R} \to \mathbb{R}$ that takes every value in every interval. That is, for every $a < b$ and every t there should exist $x \in (a, b)$ such that $f(x) = t$.

14. Let $f : \mathbb{R} \to \mathbb{R}$ be a function that has the intermediate value property: that is, if $f(a) < c < f(b)$ then there exists $x \in (a, b)$ such that $f(x) = c$. Suppose also that for every rational r the set $S_r = \{x : f(x) = r\}$ is closed. (This means that if (x_n) is any convergent sequence in S_r, then its limit also belongs to S_r.) Prove that f is continuous.

ANALYSIS 1 EXAMPLES SHEET 3

Lent Term 2015 W. T. G.

1. Suppose that $f:\mathbb{R}\to\mathbb{R}$ satisfies the inequality $|f(x)-f(y)|\le|x-y|^2$ for every $x,y\in\mathbb{R}$. Prove that f is constant.

2. (i) Let $f:\mathbb{R}\to\mathbb{R}$ be defined by $f(x)=x^2\sin(1/x)$ if $x\ne 0$ and $f(0)=0$. Prove that f is differentiable everywhere. For which x is f' continuous at x?

(ii) Give an example of a function $g:\mathbb{R}\to\mathbb{R}$ that is differentiable everywhere such that g' is not bounded on the interval $[-1,1]$.

3. Let $f:\mathbb{R}\to\mathbb{R}$ be a differentiable function with the property that $f(x)=o(x^n)$ for every positive integer n. (In other words, for every n we have $f(x)/x^n\to 0$ as $x\to 0$.) Does it follow that f is infinitely differentiable at 0?

4. By applying the mean value theorem to $\log(1+x)$ on the interval $[0,a/n]$, prove rigorously that $(1+a/n)^n\to e^a$ as $n\to\infty$.

5. Find $\lim_{n\to\infty} n(a^{1/n}-1)$, when $a>0$.

6. Let $f:\mathbb{R}\to\mathbb{R}$ be defined by $f(x)=\exp(-1/x^2)$ when $x\ne 0$ and $f(0)=0$. Prove that f is infinitely differentiable and that $f^{(n)}(0)=0$ for every $n\in\mathbb{N}$. What does Taylor's theorem tell us when we apply it to f at 0?

7. Find the radius of convergence of each of the following power series.

$$\sum_{n=0}^{\infty}\frac{2.4.6\ldots(2n+2)}{1.4.7\ldots(3n+1)}z^n \qquad \sum_{n=1}^{\infty}\frac{z^{3n}}{n2^n} \qquad \sum_{n=0}^{\infty}\frac{n^n z^n}{n!} \qquad \sum_{n=1}^{\infty} n^{\sqrt{n}}z^n$$

8. Find the derivative of $\tan x$ on the interval $(-\pi/2,\pi/2)$. How do you know that there is a differentiable inverse function $\arctan x$ from $\mathbb{R}$ to $(-\pi/2,\pi/2)$? What is its derivative? By considering derivatives, prove that $\arctan x = x - x^3/3 + x^5/5 - \ldots$ when $|x|<1$.

9. Let f and g be two functions defined and differentiable on an open interval I containing 0. Suppose that $f(0)=g(0)=0$ and that $f'(x)/g'(x)$ converges to a limit ℓ as $x\to 0$.

(i) Show that there is an open interval of the form $(0,a)$ on which g' does not vanish. Let $0<x<a$. By considering the function $F(u)=f(x)g(u)-g(x)f(u)$, prove that there exists y with $0<y<x$ such that $\frac{f'(y)}{g'(y)}=\frac{f(x)}{g(x)}$. Explain briefly why a similar statement holds for negative x.

(ii) Deduce *l'Hôpital's rule*, which states that under the conditions above, $f(x)/g(x) \to \ell$.

(iii) What is $\lim_{x\to 0}(1-\cos(\sin x))/x^2$?

10. Let (a_n) be a bounded real sequence. Prove that (a_n) has a subsequence that tends to $\limsup a_n$. What result from the course does this imply?

11. The infinite product $\prod_{n=1}^{\infty}(1+a_n)$ is said to converge to a if the sequence of partial products $P_n = (1+a_1)\dots(1+a_n)$ converges to a. Suppose that $a_n \geq 0$ for every n. Write $S_n = a_1 + \cdots + a_n$. Prove that $S_n \leq P_n \leq e^{S_n}$ for every n, and deduce that $\prod_{n=1}^{\infty}(1+a_n)$ converges if and only if $\sum_{n=1}^{\infty} a_n$ converges. Evaluate the product $\prod_{n=2}^{\infty}(1+1/(n^2-1))$.

12. Let $f:\mathbb{R}\to\mathbb{R}$ be differentiable, let a and b be real numbers with $a<b$, and suppose that $f'(a) < 0 < f'(b)$. Prove that there exists $c \in (a,b)$ such that $f'(c) = 0$. Deduce the more general result that if $f'(a) \neq f'(b)$ and z lies between $f'(a)$ and $f'(b)$, then there exists $c \in (a,b)$ such that $f'(c) = z$. (This result is called *Darboux's theorem.*)

13. Say that an ordered field $\mathbb{F}$ has the *intermediate value property* if for every $a<b$ and every continuous function $f:\mathbb{F}\to\mathbb{F}$, if $f(a)<0$ and $f(b)>0$ then there exists $c\in(a,b)$ such that $f(c) = 0$. Prove that every ordered field with the intermediate value property has the least upper bound property. (This implies that it is isomorphic to $\mathbb{R}$.)

14. (i) Show that the series $\sum_{n=1}^{\infty}\frac{z^n}{n}$ has radius of convergence 1, and that it converges for every z such that $|z|=1$, with the exception of $z=1$.

(ii) Let $z_1,\dots,z_m$ be complex numbers of modulus 1. Find a power series $\sum_{n=0}^{\infty} a_n z^n$ with radius of convergence 1 that converges for every z such that $|z|=1$, except when $z\in\{z_1,\dots,z_m\}$, when it diverges.

15. (i) Let f and g be two n-times-differentiable functions from $\mathbb{R}$ to $\mathbb{R}$. For $k\leq n$ and $x\in\mathbb{R}$, say that f and g *agree to order k at x* if $f^{(j)}(x) = g^{(j)}(x)$ for $j=0,1,\dots,k-1$. Let $x_1<x_2<\cdots<x_r$ be real numbers, let $k_1,\dots,k_r$ be non-negative integers such that $k_1+\cdots+k_r = n$, and suppose that for each $i\leq r$ the functions f and g agree to order k_i at x_i. If $r\geq 2$, prove that there exists x in the open interval (x_1,x_r) such that $f^{(n-1)}(x) = g^{(n-1)}(x)$. [Note that if you can do this when g is the zero function then you can do it in general. If you still find it too hard, then try it in the case $r=n$, so $k_1=\cdots=k_n=1$, and in the case $k=2$, to get an idea what is going on.]

(ii) Let f be n-times differentiable, let $x_1<\cdots<x_r$ be real numbers and let $k_1,\dots,k_r$ be non-negative integers with $k_1+\cdots+k_r=n$. Prove that there is a polynomial p of degree at most $n-1$ such that for every $i\leq r$ and every $j<k_i$ we have $p^{(j)}(x_i) = f^{(j)}(x_i)$. [Hint: start by building a suitable basis of polynomials and then take linear combinations.]

(iii) Find an expression for the constant value of $p^{(n-1)}$.

ANALYSIS 1 EXAMPLES SHEET 4

Lent Term 2015 W. T. G.

1. Show directly from the definition of an integral that $\int_0^a x^2 dx = a^3/3$ for $a > 0$.

2. Give an example of a continuous function $f : [0, \infty) \to [0, \infty)$ such that $\int_0^\infty f(x)dx$ exists but f is unbounded.

3. Give an example of an integrable function $f : [0, 1] \to \mathbb{R}$ such that $f(x) \geq 0$ for every x, $f(y) > 0$ for some y, and $\int_0^1 f(x)dx = 0$.

Prove that this cannot happen if in addition f is continuous.

4. Let $f : \mathbb{R} \to \mathbb{R}$ be monotonic. Show that the set of x such that f is discontinuous at x is countable.

Let (x_n) be a sequence of distinct points in $(0, 1]$. Let $f_n(x) = 0$ if $0 \leq x < x_n$ and let $f_n(x) = 1$ if $x_n \leq x \leq 1$. For each x, let $f(x) = \sum_{n=1}^\infty 2^{-n} f_n(x)$. Prove that this series converges for every $x \in [0, 1]$.

Explain why f must be integrable.

Prove that f is discontinuous at every x_n.

5. Define a function $f : [0, 1] \to \mathbb{R}$ as follows. If x is irrational, then $f(x) = 0$. If x is rational, then write it in its lowest terms as p/q and then $f(x) = 1/q$. Prove that f is integrable. What is $\int_0^1 f(x)dx$?

6. Let $a < b$ and let $f : [a, b] \to \mathbb{R}$ be a Riemann integrable function such that $f(x) \geq 0$ for every x. Prove that if $\int_a^b f(x)dx = 0$, then for every closed subinterval $I \subset [a, b]$ of positive length and every $\epsilon > 0$ there exists a closed subinterval $J \subset I$ of positive length such that $f(x) \leq \epsilon$ for every $x \in J$.

Deduce that if $f(x) > 0$ for every x, then $\int_a^b f(x)dx > 0$.

7. Do these improper integrals converge?

(i) $\int_1^\infty \sin^2(1/x)dx$.

(ii) $\int_0^\infty x^p \exp(-x^q)dx$ (with $p, q > 0$).

(iii) $\int_0^\infty \sin(x^2)dx$.

8. Prove that $\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n}\to\log 2$ as $n\to\infty$, and find the limit of $\frac{1}{n+1}-\frac{1}{n+2}+\cdots+\frac{(-1)^{n-1}}{2n}$.

9. Let $f:[a,b]\to\mathbb{R}$ be continuous and suppose that $\int_a^b f(x)g(x)dx=0$ for every continuous function $g:[a,b]\to\mathbb{R}$ with $g(a)=g(b)=0$. Must f vanish identically?

10. Let $f:[0,1]\to\mathbb{R}$ be continuous. Let $G(x,t)=t(x-1)$ when $t\le x$ and $x(t-1)$ when $t\ge x$. Let $g(x)=\int_0^1 f(t)G(x,t)dt$. Show that $g''(x)$ exists for $x\in(0,1)$ and equals $f(x)$.

11. For positive x, define $L(x)$ to be $\int_1^x \frac{dt}{t}$. Prove directly from this definition that the function L has the properties one normally expects of the logarithm function. In particular, prove that $L(ab)=L(a)+L(b)$ for all positive a and b. If you adopted this as your fundamental definition of natural logarithms, then how would you define e?

12. For each non-negative integer n let $I_n(\theta)=\int_{-1}^1(1-x^2)^n\cos(\theta x)dx$. Prove that $\theta^2 I_n = 2n(2n-1)I_{n-1}-4n(n-1)I_{n-2}$ for all $n\ge 2$, and hence that $\theta^{2n+1}I_n(\theta)=n!(P_n(\theta)\sin\theta+Q_n(\theta)\cos\theta)$ for some pair P_n and Q_n of polynomials of degree at most $2n$ with integer coefficients.

Deduce that π is irrational.

13. Let $f:[-1,1]$ be defined by $f(x)=x\sin(1/x)$ when $x\ne 0$ and $f(0)=0$. Explain why f is integrable. Prove that there do not exist increasing functions g and h, defined on $[-1,1]$, such that $f(x)=g(x)-h(x)$ for every x.

14. Prove that if $f:[0,1]\to\mathbb{R}$ is integrable, then f has infinitely many points of continuity.

15*. Let $f:[0,1]\to\mathbb{R}$ be a function that is differentiable everywhere (with right and left derivatives at the end points) with a derivative f' that is bounded. Must f' be integrable?

Comments and corrections to wtg10@dpmms.cam.ac.uk