Part IA — Analysis I
Definitions

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Limits and convergence
Sequences and series in $\mathbb{R}$ and $\mathbb{C}$. Sums, products and quotients. Absolute convergence; absolute convergence implies convergence. The Bolzano-Weierstrass theorem and applications (the General Principle of Convergence). Comparison and ratio tests, alternating series test. [6]

Continuity
Continuity of real- and complex-valued functions defined on subsets of $\mathbb{R}$ and $\mathbb{C}$. The intermediate value theorem. A continuous function on a closed bounded interval is bounded and attains its bounds. [3]

Differentiability
Differentiability of functions from $\mathbb{R}$ to $\mathbb{R}$. Derivative of sums and products. The chain rule. Derivative of the inverse function. Rolle's theorem; the mean value theorem. One-dimensional version of the inverse function theorem. Taylor's theorem from $\mathbb{R}$ to $\mathbb{R}$; Lagrange's form of the remainder. Complex differentiation. [5]

Power series
Complex power series and radius of convergence. Exponential, trigonometric and hyperbolic functions, and relations between them. *Direct proof of the differentiability of a power series within its circle of convergence*. [4]

Integration
Definition and basic properties of the Riemann integral. A non-integrable function. Integrability of monotonic functions. Integrability of piecewise-continuous functions. The fundamental theorem of calculus. Differentiation of indefinite integrals. Integration by parts. The integral form of the remainder in Taylor's theorem. Improper integrals. [6]

Contents

0 Introduction

1 The real number system

Definition (Field)**.** A *field* is a set $\mathbb{F}$ with two binary operations $+$ and $\times$ that satisfies all the familiar properties satisfied by addition and multiplication in $\mathbb{Q}$, namely

(i) $(\forall a, b, c)\, a + (b + c) = (a + b) + c$ and $a \times (b \times c) = (a \times b) \times c$ (associativity)

(ii) $(\forall a, b)\, a + b = b + a$ and $a \times b = b \times a$ (commutativity)

(iii) $\exists 0, 1$ such that $(\forall a) a + 0 = a$ and $a \times 1 = a$ (identity)

(iv) $\forall a\, \exists(-a)$ such that $a + (-a) = 0$. If $a \neq 0$, then $\exists a^{-1}$ such that $a \times a^{-1} = 1$. (inverses)

(v) $(\forall a, b, c)\, a \times (b + c) = (a \times b) + (a \times c)$ (distributivity)

Definition (Totally ordered set)**.** An *(totally) ordered set* is a set X with a relation $<$ that satisfies

(i) If $x, y, z \in X$, $x < y$ and $y < z$, then $x < z$ (transitivity)

(ii) If $x, y \in X$, exactly one of $x < y, x = y, y < x$ holds (trichotomy)

Definition (Ordered field)**.** An *ordered field* is a field $\mathbb{F}$ with a relation $<$ that makes $\mathbb{F}$ into an ordered set such that

(i) if $x, y, z \in \mathbb{F}$ and $x < y$, then $x + z < y + z$

(ii) if $x, y, z \in \mathbb{F}$, $x < y$ and $z > 0$, then $xz < yz$

Definition (Least upper bound)**.** Let X be an ordered set and let $A \subseteq X$. An *upper bound* for A is an element $x \in X$ such that $(\forall a \in A)\, a \leq x$. If A has an upper bound, then we say that A is *bounded above*.

An upper bound x for A is a *least upper bound* or *supremum* if nothing smaller that x is an upper bound. That is, we need

(i) $(\forall a \in A)\, a \leq x$

(ii) $(\forall y < x)(\exists a \in A)\, a > y$

We usually write $\sup A$ for the supremum of A when it exists. If $\sup A \in A$, then we call it $\max A$, the maximum of A.

Definition (Least upper bound property)**.** An ordered set X has the *least upper bound property* if every non-empty subset of X that is bounded above has a supremum.

Definition (Real numbers)**.** The *real numbers* is an ordered field with the least upper bound property.

2 Convergence of sequences

2.1 Definitions

Definition (Sequence)**.** A *sequence* is, formally, a function $a : \mathbb{N} \to \mathbb{R}$ (or $\mathbb{C}$). Usually (i.e. always), we write a_n instead of $a(n)$. Instead of a, we usually write it as (a_n), $(a_n)_1^\infty$ or $(a_n)_{n=1}^\infty$ to indicate it is a sequence.

Definition (Convergence of sequence)**.** Let (a_n) be a sequence and $\ell \in \mathbb{R}$. Then a_n *converges to* ℓ, *tends to* ℓ, or $a_n \to \ell$, if for all $\varepsilon > 0$, there is some $N \in \mathbb{N}$ such that whenever $n > N$, we have $|a_n - \ell| < \varepsilon$. In symbols, this says

$$(\forall \varepsilon > 0)(\exists N)(\forall n \geq N)\, |a_n - \ell| < \varepsilon.$$

We say ℓ is the *limit* of (a_n).

Definition (Bounded sequence)**.** A sequence (a_n) is *bounded* if

$$(\exists C)(\forall n)\, |a_n| \leq C.$$

A sequence is *eventually bounded* if

$$(\exists C)(\exists N)(\forall n \geq N)\, |a_n| \leq C.$$

2.2 Sums, products and quotients

2.3 Monotone-sequences property

Definition (Monotone sequence)**.** A sequence (a_n) is *increasing* if $a_n \leq a_{n+1}$ for all n.

It is *strictly increasing* if $a_n < a_{n+1}$ for all n. *(Strictly) decreasing* sequences are defined analogously.

A sequence is *(strictly) monotone* if it is (strictly) increasing or (strictly) decreasing.

Definition (Monotone-sequences property)**.** An ordered field has the *monotone sequences property* if every increasing sequence that is bounded above converges.

Definition (Subsequence)**.** Let (a_n) be a sequence. A *subsequence* of (a_n) is a sequence of the form $a_{n_1}, a_{n_2}, \cdots$, where $n_1 < n_2 < \cdots$.

2.4 Cauchy sequences

Definition (Cauchy sequence)**.** A sequence (a_n) is *Cauchy* if for all ε, there is some $N \in \mathbb{N}$ such that whenever $p, q \geq N$, we have $|a_p - a_q| < \varepsilon$. In symbols, we have

$$(\forall \varepsilon > 0)(\exists N)(\forall p, q \geq N)\, |a_p - a_q| < \varepsilon.$$

Definition (Complete ordered field)**.** An ordered field in which every Cauchy sequence converges is called *complete*.

2.5 Limit supremum and infimum

Definition (Limit supremum/infimum)**.** Let (a_n) be a bounded sequence. We define the *limit supremum* as

$$\limsup_{n\to\infty} a_n = \lim_{n\to\infty}\left(\sup_{m\geq n} a_m\right).$$

To see that this exists, set $b_n = \sup_{m\geq n} a_m$. Then (b_n) is decreasing since we are taking the supremum of fewer and fewer things, and is bounded below by any lower bound for (a_n) since $b_n \geq a_n$. So it converges.

Similarly, we define the *limit infimum* as

$$\liminf_{n\to\infty} a_n = \lim_{n\to\infty}\left(\inf_{m\geq n} a_m\right).$$

3 Convergence of infinite sums

3.1 Infinite sums

Definition (Convergence of infinite sums and partial sums)**.** Let (a_n) be a real sequence. For each N, define

$$S_N = \sum_{n=1}^{N} a_n.$$

If the sequence (S_N) converges to some limit s, then we say that

$$\sum_{n=1}^{\infty} a_n = s,$$

and we say that the series $\displaystyle\sum_{n=1}^{\infty} a_n$ *converges*.

We call S_N the Nth *partial sum*.

3.2 Absolute convergence

Definition (Absolute convergence)**.** A series $\sum a_n$ *converges absolutely* if the series $\sum |a_n|$ converges.

Definition (Unconditional convergence)**.** A series $\sum a_n$ *converges unconditionally* if the series $\sum_{n=1}^{\infty} a_{\pi(n)}$ converges for every bijection $\pi : \mathbb{N} \to \mathbb{N}$, i.e. no matter how we re-order the elements of a_n, the sum still converges.

3.3 Convergence tests

3.4 Complex versions

4 Continuous functions

4.1 Continuous functions

Definition (Continuous function)**.** Let $A \subseteq \mathbb{R}$, $a \in A$, and $f : A \to \mathbb{R}$. Then f is *continuous at* a if for any $\varepsilon > 0$, there is some $\delta > 0$ such that if $y \in A$ is such that $|y - a| < \delta$, then $|f(y) - f(a)| < \varepsilon$. In symbols, we have

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall y \in A)\, |y - a| < \delta \Rightarrow |f(y) - f(a)| < \varepsilon.$$

f is *continuous* if it is continuous at every $a \in A$. In symbols, we have

$$(\forall a \in A)(\forall \varepsilon > 0)(\exists \delta > 0)(\forall y \in A)\, |y - a| < \delta \Rightarrow |f(y) - f(a)| < \varepsilon.$$

4.2 Continuous induction*

Definition (Cover of a set)**.** Let $A \subseteq \mathbb{R}$. A *cover* of A by open intervals is a set $\{I_\gamma : \gamma \in \Gamma\}$ where each I_γ is an open interval and $A \subseteq \bigcup_{\gamma \in \Gamma} I_\gamma$.

A *finite subcover* is a finite subset $\{I_{\gamma_1}, \cdots, I_{\gamma_n}\}$ of the cover that is still a cover.

5 Differentiability

5.1 Limits

Definition (Limit of functions)**.** Let $A \subseteq \mathbb{R}$ and let $f : A \to \mathbb{R}$. We say

$$\lim_{x \to a} f(x) = \ell,$$

or "$f(x) \to \ell$ as $x \to a$", if

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x \in A)\, 0 < |x - a| < \delta \Rightarrow |f(x) - \ell| < \varepsilon.$$

We couldn't care less what happens when $x = a$, hence the strict inequality $0 < |x - a|$. In fact, f doesn't even have to be defined at $x = a$.

5.2 Differentiation

Definition (Differentiable function)**.** f is *differentiable* at a with derivative λ if

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lambda.$$

Equivalently, if

$$\lim_{h \to 0} \frac{f(a + h) - f(a)}{h} = \lambda.$$

We write $\lambda = f'(a)$.

Definition (Multiple derivatives)**.** This is defined recursively: f is $(n + 1)$-times differentiable if it is n-times differentiable and its nth derivative $f^{(n)}$ is differentiable. We write $f^{(n+1)}$ for the derivative of $f^{(n)}$, i.e. the $(n + 1)$th derivative of f.

Informally, we will say f is n-times differentiable if we can differentiate it n times, and the nth derivative is $f^{(n)}$.

5.3 Differentiation theorems

5.4 Complex differentiation

Definition (Complex differentiability)**.** Let $f : \mathbb{C} \to \mathbb{C}$. Then f is differentiable at z with derivative $f'(z)$ if

$$\lim_{h \to 0} \frac{f(z + h) - f(z)}{h} \text{ exists and equals } f'(z).$$

Equivalently,

$$f(z + h) = f(z) + hf'(z) + o(h).$$

6 Complex power series

Definition (Complex power series). A *complex power series* is a series of the form
$$\sum_{n=0}^{\infty} a_n z^n.$$
when $z \in \mathbb{C}$ and $a_n \in \mathbb{C}$ for all n. When it converges, it is a function of z.

Definition (Radius of convergence). The *radius of convergence* of a power series $\sum a_n z^n$ is
$$R = \sup\left\{|z| : \sum a_n z^n \text{ converges }\right\}.$$
$\{z : |z| < R\}$ is called the *circle of convergence*.[1].

If $|z| < R$, then $\sum a_n z^n$ converges. If $|z| > R$, then $\sum a_n z^n$ diverges. When $|z| = R$, the series can converge at some points and not the others.

6.1 Exponential and trigonometric functions

Definition (Exponential function). The *exponential function* is
$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$
By the ratio test, this converges on all of $\mathbb{C}$.

Definition (Sine and cosine).
$$\begin{aligned}\sin z &= \frac{e^{iz} - e^{-iz}}{2i} = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots\\ \cos z &= \frac{e^{iz} + e^{-iz}}{2} = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots.\end{aligned}$$

Definition (Pi). Define the smallest x such that $\cos x = 0$ to be $\frac{\pi}{2}$.

6.2 Differentiating power series

6.3 Hyperbolic trigonometric functions

Definition (Hyperbolic sine and cosine). We define
$$\begin{aligned}\cosh z &= \frac{e^z + e^{-z}}{2} = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \frac{z^6}{6!} + \cdots\\ \sinh z &= \frac{e^z - e^{-z}}{2} = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \frac{z^7}{7!} + \cdots\end{aligned}$$

[1]Note to pedants: yes it is a disc, not a circle

7 The Riemann Integral

7.1 Riemann Integral

Definition (Dissections)**.** Let $[a, b]$ be a closed interval. A *dissection* of $[a, b]$ is a sequence $a = x_0 < x_1 < x_2 < \cdots < x_n = b$.

Definition (Upper and lower sums)**.** Given a dissection $\mathcal{D}$, the *upper sum* and *lower sum* are defined by the formulae

$$U_{\mathcal{D}}(f) = \sum_{i=1}^{n}(x_i - x_{i-1}) \sup_{x \in [x_{i-1}, x_i]} f(x)$$
$$L_{\mathcal{D}}(f) = \sum_{i=1}^{n}(x_i - x_{i-1}) \inf_{x \in [x_{i-1}, x_i]} f(x)$$

Sometimes we use the shorthand

$$M_i = \sup_{x \in [x_{i-1}, x_i]} f(x), \quad m_i = \inf_{x \in [x_{i-1} - x_i]} f(x).$$

Definition (Refining dissections)**.** If $\mathcal{D}_1$ and $\mathcal{D}_2$ are dissections of $[a, b]$, we say that $\mathcal{D}_2$ refines $\mathcal{D}_1$ if every point of $\mathcal{D}_1$ is a point of $\mathcal{D}_2$.

Definition (Least common refinement)**.** If $\mathcal{D}_1$ and $\mathcal{D}_2$ be dissections of $[a, b]$. Then the least common refinement of $\mathcal{D}_1$ and $\mathcal{D}_2$ is the dissection made out of the points of $\mathcal{D}_1$ and $\mathcal{D}_2$.

Definition (Upper, lower, and Riemann integral)**.** The *upper integral* is

$$\overline{\int_a^b} f(x)\ \mathrm{d}x = \inf_{\mathcal{D}} U_{\mathcal{D}} f.$$

The *lower integral* is

$$\underline{\int_a^b} f(x)\ \mathrm{d}x = \sup_{\mathcal{D}} L_{\mathcal{D}} f.$$

If these are equal, then we call their common value the *Riemann integral* of f, and is denoted $\int_a^b f(x)\ \mathrm{d}x$.

If this exists, we say f is *Riemann integrable*.

Definition (Mesh)**.** The *mesh* of a dissection $\mathcal{D}$ is $\max_i(x_{i+1} - x_i)$.

Definition (Uniform continuity*)**.** Let $A \subseteq \mathbb{R}$ and let $f : A \to \mathbb{R}$. Then f is *uniformly continuous* if

$$(\forall \varepsilon)(\exists \delta > 0)(\forall x)(\forall y)\ |x - y| < \delta \Rightarrow |f(x) - f(y)| \leq \varepsilon.$$

Notation. If $b > a$, we define

$$\int_b^a f(x)\ \mathrm{d}x = -\int_a^b f(x)\ \mathrm{d}x.$$

7.2 Improper integrals

Definition (Improper integral)**.** Suppose that we have a function $f : [a, b] \to \mathbb{R}$ such that, for every $\varepsilon > 0$, f is integrable on $[a + \varepsilon, b]$ and $\lim_{\varepsilon\to 0} \int_{a+\varepsilon}^{b} f(x)\ \mathrm{d}x$ exists. Then we define the improper integral

$$\int_a^b f(x)\ \mathrm{d}x \text{ to be } \lim_{\varepsilon\to 0} \int_{a+\varepsilon}^{b} f(x)\ \mathrm{d}x.$$

even if the Riemann integral does not exist.

We can do similarly for $[a, b - \varepsilon]$, or integral to infinity:

$$\int_a^\infty f(x)\ \mathrm{d}x = \lim_{b\to\infty} \int_a^b f(x)\ \mathrm{d}x.$$

when it exists.