

# Part IA — Analysis I

## Definitions

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Lent 2015

These notes are not endorsed by the lecturers, and I have modified them (often significantly) after lectures. They are nowhere near accurate representations of what was actually lectured, and in particular, all errors are almost surely mine.

### **Limits and convergence**

Sequences and series in  $\mathbb{R}$  and  $\mathbb{C}$ . Sums, products and quotients. Absolute convergence; absolute convergence implies convergence. The Bolzano-Weierstrass theorem and applications (the General Principle of Convergence). Comparison and ratio tests, alternating series test. [6]

### **Continuity**

Continuity of real- and complex-valued functions defined on subsets of  $\mathbb{R}$  and  $\mathbb{C}$ . The intermediate value theorem. A continuous function on a closed bounded interval is bounded and attains its bounds. [3]

### **Differentiability**

Differentiability of functions from  $\mathbb{R}$  to  $\mathbb{R}$ . Derivative of sums and products. The chain rule. Derivative of the inverse function. Rolle's theorem; the mean value theorem. One-dimensional version of the inverse function theorem. Taylor's theorem from  $\mathbb{R}$  to  $\mathbb{R}$ ; Lagrange's form of the remainder. Complex differentiation. [5]

### **Power series**

Complex power series and radius of convergence. Exponential, trigonometric and hyperbolic functions, and relations between them. \*Direct proof of the differentiability of a power series within its circle of convergence\*. [4]

### **Integration**

Definition and basic properties of the Riemann integral. A non-integrable function. Integrability of monotonic functions. Integrability of piecewise-continuous functions. The fundamental theorem of calculus. Differentiation of indefinite integrals. Integration by parts. The integral form of the remainder in Taylor's theorem. Improper integrals. [6]

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## 0 Introduction

## 1 The real number system

**Definition (Field).** A *field* is a set  $\mathbb{F}$  with two binary operations  $+$  and  $\times$  that satisfies all the familiar properties satisfied by addition and multiplication in  $\mathbb{Q}$ , namely

- (i)  $(\forall a, b, c) a + (b + c) = (a + b) + c$  and  $a \times (b \times c) = (a \times b) \times c$  (associativity)
- (ii)  $(\forall a, b) a + b = b + a$  and  $a \times b = b \times a$  (commutativity)
- (iii)  $\exists 0, 1$  such that  $(\forall a) a + 0 = a$  and  $a \times 1 = a$  (identity)
- (iv)  $\forall a \exists (-a)$  such that  $a + (-a) = 0$ . If  $a \neq 0$ , then  $\exists a^{-1}$  such that  $a \times a^{-1} = 1$ . (inverses)
- (v)  $(\forall a, b, c) a \times (b + c) = (a \times b) + (a \times c)$  (distributivity)

**Definition (Totally ordered set).** An (*totally*) *ordered set* is a set  $X$  with a relation  $<$  that satisfies

- (i) If  $x, y, z \in X$ ,  $x < y$  and  $y < z$ , then  $x < z$  (transitivity)
- (ii) If  $x, y \in X$ , exactly one of  $x < y, x = y, y < x$  holds (trichotomy)

**Definition (Ordered field).** An *ordered field* is a field  $\mathbb{F}$  with a relation  $<$  that makes  $\mathbb{F}$  into an ordered set such that

- (i) if  $x, y, z \in \mathbb{F}$  and  $x < y$ , then  $x + z < y + z$
- (ii) if  $x, y, z \in \mathbb{F}$ ,  $x < y$  and  $z > 0$ , then  $xz < yz$

**Definition (Least upper bound).** Let  $X$  be an ordered set and let  $A \subseteq X$ . An *upper bound* for  $A$  is an element  $x \in X$  such that  $(\forall a \in A) a \leq x$ . If  $A$  has an upper bound, then we say that  $A$  is *bounded above*.

An upper bound  $x$  for  $A$  is a *least upper bound* or *supremum* if nothing smaller than  $x$  is an upper bound. That is, we need

- (i)  $(\forall a \in A) a \leq x$
- (ii)  $(\forall y < x)(\exists a \in A) a \geq y$

We usually write  $\sup A$  for the supremum of  $A$  when it exists. If  $\sup A \in A$ , then we call it  $\max A$ , the maximum of  $A$ .

**Definition (Least upper bound property).** An ordered set  $X$  has the *least upper bound property* if every non-empty subset of  $X$  that is bounded above has a supremum.

**Definition (Real numbers).** The *real numbers* is an ordered field with the least upper bound property.

## 2 Convergence of sequences

### 2.1 Definitions

**Definition** (Sequence). A *sequence* is, formally, a function  $a : \mathbb{N} \rightarrow \mathbb{R}$  (or  $\mathbb{C}$ ). Usually (i.e. always), we write  $a_n$  instead of  $a(n)$ . Instead of  $a$ , we usually write it as  $(a_n)$ ,  $(a_n)_1^\infty$  or  $(a_n)_{n=1}^\infty$  to indicate it is a sequence.

**Definition** (Convergence of sequence). Let  $(a_n)$  be a sequence and  $\ell \in \mathbb{R}$ . Then  $a_n$  *converges to*  $\ell$ , *tends to*  $\ell$ , or  $a_n \rightarrow \ell$ , if for all  $\varepsilon > 0$ , there is some  $N \in \mathbb{N}$  such that whenever  $n > N$ , we have  $|a_n - \ell| < \varepsilon$ . In symbols, this says

$$(\forall \varepsilon > 0)(\exists N)(\forall n \geq N) |a_n - \ell| < \varepsilon.$$

We say  $\ell$  is the *limit* of  $(a_n)$ .

**Definition** (Bounded sequence). A sequence  $(a_n)$  is *bounded* if

$$(\exists C)(\forall n) |a_n| \leq C.$$

A sequence is *eventually bounded* if

$$(\exists C)(\exists N)(\forall n \geq N) |a_n| \leq C.$$

### 2.2 Sums, products and quotients

### 2.3 Monotone-sequences property

**Definition** (Monotone sequence). A sequence  $(a_n)$  is *increasing* if  $a_n \leq a_{n+1}$  for all  $n$ .

It is *strictly increasing* if  $a_n < a_{n+1}$  for all  $n$ . (*Strictly decreasing* sequences are defined analogously.

A sequence is (*strictly*) *monotone* if it is (strictly) increasing or (strictly) decreasing.

**Definition** (Monotone-sequences property). An ordered field has the *monotone sequences property* if every increasing sequence that is bounded above converges.

**Definition** (Subsequence). Let  $(a_n)$  be a sequence. A *subsequence* of  $(a_n)$  is a sequence of the form  $a_{n_1}, a_{n_2}, \dots$ , where  $n_1 < n_2 < \dots$ .

### 2.4 Cauchy sequences

**Definition** (Cauchy sequence). A sequence  $(a_n)$  is *Cauchy* if for all  $\varepsilon$ , there is some  $N \in \mathbb{N}$  such that whenever  $p, q \geq N$ , we have  $|a_p - a_q| < \varepsilon$ . In symbols, we have

$$(\forall \varepsilon > 0)(\exists N)(\forall p, q \geq N) |a_p - a_q| < \varepsilon.$$

**Definition** (Complete ordered field). An ordered field in which every Cauchy sequence converges is called *complete*.

## 2.5 Limit supremum and infimum

**Definition** (Limit supremum/infimum). Let  $(a_n)$  be a bounded sequence. We define the *limit supremum* as

$$\limsup_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \left( \sup_{m \geq n} a_m \right).$$

To see that this exists, set  $b_n = \sup_{m \geq n} a_m$ . Then  $(b_n)$  is decreasing since we are taking the supremum of fewer and fewer things, and is bounded below by any lower bound for  $(a_n)$  since  $b_n \geq a_n$ . So it converges.

Similarly, we define the *limit infimum* as

$$\liminf_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \left( \inf_{m \geq n} a_m \right).$$

## 3 Convergence of infinite sums

### 3.1 Infinite sums

**Definition** (Convergence of infinite sums and partial sums). Let  $(a_n)$  be a real sequence. For each  $N$ , define

$$S_N = \sum_{n=1}^N a_n.$$

If the sequence  $(S_N)$  converges to some limit  $s$ , then we say that

$$\sum_{n=1}^{\infty} a_n = s,$$

and we say that the series  $\sum_{n=1}^{\infty} a_n$  *converges*.

We call  $S_N$  the  $N$ th *partial sum*.

### 3.2 Absolute convergence

**Definition** (Absolute convergence). A series  $\sum a_n$  *converges absolutely* if the series  $\sum |a_n|$  converges.

**Definition** (Unconditional convergence). A series  $\sum a_n$  *converges unconditionally* if the series  $\sum_{n=1}^{\infty} a_{\pi(n)}$  converges for every bijection  $\pi : \mathbb{N} \rightarrow \mathbb{N}$ , i.e. no matter how we re-order the elements of  $a_n$ , the sum still converges.

### 3.3 Convergence tests

### 3.4 Complex versions

## 4 Continuous functions

### 4.1 Continuous functions

**Definition** (Continuous function). Let  $A \subseteq \mathbb{R}$ ,  $a \in A$ , and  $f : A \rightarrow \mathbb{R}$ . Then  $f$  is *continuous at  $a$*  if for any  $\varepsilon > 0$ , there is some  $\delta > 0$  such that if  $y \in A$  is such that  $|y - a| < \delta$ , then  $|f(y) - f(a)| < \varepsilon$ . In symbols, we have

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall y \in A) |y - a| < \delta \Rightarrow |f(y) - f(a)| < \varepsilon.$$

$f$  is *continuous* if it is continuous at every  $a \in A$ . In symbols, we have

$$(\forall a \in A)(\forall \varepsilon > 0)(\exists \delta > 0)(\forall y \in A) |y - a| < \delta \Rightarrow |f(y) - f(a)| < \varepsilon.$$

### 4.2 Continuous induction\*

**Definition** (Cover of a set). Let  $A \subseteq \mathbb{R}$ . A *cover* of  $A$  by open intervals is a set  $\{I_\gamma : \gamma \in \Gamma\}$  where each  $I_\gamma$  is an open interval and  $A \subseteq \bigcup_{\gamma \in \Gamma} I_\gamma$ .

A *finite subcover* is a finite subset  $\{I_{\gamma_1}, \dots, I_{\gamma_n}\}$  of the cover that is still a cover.

## 5 Differentiability

### 5.1 Limits

**Definition** (Limit of functions). Let  $A \subseteq \mathbb{R}$  and let  $f : A \rightarrow \mathbb{R}$ . We say

$$\lim_{x \rightarrow a} f(x) = \ell,$$

or “ $f(x) \rightarrow \ell$  as  $x \rightarrow a$ ”, if

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x \in A) 0 < |x - a| < \delta \Rightarrow |f(x) - \ell| < \varepsilon.$$

We couldn't care less what happens when  $x = a$ , hence the strict inequality  $0 < |x - a|$ . In fact,  $f$  doesn't even have to be defined at  $x = a$ .

### 5.2 Differentiation

**Definition** (Differentiable function).  $f$  is *differentiable* at  $a$  with derivative  $\lambda$  if

$$\lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} = \lambda.$$

Equivalently, if

$$\lim_{h \rightarrow 0} \frac{f(a + h) - f(a)}{h} = \lambda.$$

We write  $\lambda = f'(a)$ .

**Definition** (Multiple derivatives). This is defined recursively:  $f$  is  $(n + 1)$ -times differentiable if it is  $n$ -times differentiable and its  $n$ th derivative  $f^{(n)}$  is differentiable. We write  $f^{(n+1)}$  for the derivative of  $f^{(n)}$ , i.e. the  $(n + 1)$ th derivative of  $f$ .

Informally, we will say  $f$  is  $n$ -times differentiable if we can differentiate it  $n$  times, and the  $n$ th derivative is  $f^{(n)}$ .

### 5.3 Differentiation theorems

### 5.4 Complex differentiation

**Definition** (Complex differentiability). Let  $f : \mathbb{C} \rightarrow \mathbb{C}$ . Then  $f$  is differentiable at  $z$  with derivative  $f'(z)$  if

$$\lim_{h \rightarrow 0} \frac{f(z + h) - f(z)}{h} \text{ exists and equals } f'(z).$$

Equivalently,

$$f(z + h) = f(z) + hf'(z) + o(h).$$

## 6 Complex power series

**Definition** (Complex power series). A *complex power series* is a series of the form

$$\sum_{n=0}^{\infty} a_n z^n.$$

when  $z \in \mathbb{C}$  and  $a_n \in \mathbb{C}$  for all  $n$ . When it converges, it is a function of  $z$ .

**Definition** (Radius of convergence). The *radius of convergence* of a power series  $\sum a_n z^n$  is

$$R = \sup \left\{ |z| : \sum a_n z^n \text{ converges} \right\}.$$

$\{z : |z| < R\}$  is called the *circle of convergence*.<sup>1</sup>

If  $|z| < R$ , then  $\sum a_n z^n$  converges. If  $|z| > R$ , then  $\sum a_n z^n$  diverges. When  $|z| = R$ , the series can converge at some points and not the others.

### 6.1 Exponential and trigonometric functions

**Definition** (Exponential function). The *exponential function* is

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

By the ratio test, this converges on all of  $\mathbb{C}$ .

**Definition** (Sine and cosine).

$$\begin{aligned} \sin z &= \frac{e^{iz} - e^{-iz}}{2i} = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots \\ \cos z &= \frac{e^{iz} + e^{-iz}}{2} = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots \end{aligned}$$

**Definition** (Pi). Define the smallest  $x$  such that  $\cos x = 0$  to be  $\frac{\pi}{2}$ .

### 6.2 Differentiating power series

### 6.3 Hyperbolic trigonometric functions

**Definition** (Hyperbolic sine and cosine). We define

$$\begin{aligned} \cosh z &= \frac{e^z + e^{-z}}{2} = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \frac{z^6}{6!} + \cdots \\ \sinh z &= \frac{e^z - e^{-z}}{2} = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \frac{z^7}{7!} + \cdots \end{aligned}$$

<sup>1</sup>Note to pedants: yes it is a disc, not a circle

## 7 The Riemann Integral

### 7.1 Riemann Integral

**Definition** (Dissections). Let  $[a, b]$  be a closed interval. A *dissection* of  $[a, b]$  is a sequence  $a = x_0 < x_1 < x_2 < \cdots < x_n = b$ .

**Definition** (Upper and lower sums). Given a dissection  $\mathcal{D}$ , the *upper sum* and *lower sum* are defined by the formulae

$$U_{\mathcal{D}}(f) = \sum_{i=1}^n (x_i - x_{i-1}) \sup_{x \in [x_{i-1}, x_i]} f(x)$$

$$L_{\mathcal{D}}(f) = \sum_{i=1}^n (x_i - x_{i-1}) \inf_{x \in [x_{i-1}, x_i]} f(x)$$

Sometimes we use the shorthand

$$M_i = \sup_{x \in [x_{i-1}, x_i]} f(x), \quad m_i = \inf_{x \in [x_{i-1}, x_i]} f(x).$$

**Definition** (Refining dissections). If  $\mathcal{D}_1$  and  $\mathcal{D}_2$  are dissections of  $[a, b]$ , we say that  $\mathcal{D}_2$  *refines*  $\mathcal{D}_1$  if every point of  $\mathcal{D}_1$  is a point of  $\mathcal{D}_2$ .

**Definition** (Least common refinement). If  $\mathcal{D}_1$  and  $\mathcal{D}_2$  be dissections of  $[a, b]$ . Then the least common refinement of  $\mathcal{D}_1$  and  $\mathcal{D}_2$  is the dissection made out of the points of  $\mathcal{D}_1$  and  $\mathcal{D}_2$ .

**Definition** (Upper, lower, and Riemann integral). The *upper integral* is

$$\overline{\int_a^b} f(x) \, dx = \inf_{\mathcal{D}} U_{\mathcal{D}} f.$$

The *lower integral* is

$$\underline{\int_a^b} f(x) \, dx = \sup_{\mathcal{D}} L_{\mathcal{D}} f.$$

If these are equal, then we call their common value the *Riemann integral* of  $f$ , and is denoted  $\int_a^b f(x) \, dx$ .

If this exists, we say  $f$  is *Riemann integrable*.

**Definition** (Mesh). The *mesh* of a dissection  $\mathcal{D}$  is  $\max_i (x_{i+1} - x_i)$ .

**Definition** (Uniform continuity\*). Let  $A \subseteq \mathbb{R}$  and let  $f : A \rightarrow \mathbb{R}$ . Then  $f$  is *uniformly continuous* if

$$(\forall \varepsilon)(\exists \delta > 0)(\forall x)(\forall y) |x - y| < \delta \Rightarrow |f(x) - f(y)| \leq \varepsilon.$$

**Notation.** If  $b > a$ , we define

$$\int_b^a f(x) \, dx = - \int_a^b f(x) \, dx.$$

## 7.2 Improper integrals

**Definition** (Improper integral). Suppose that we have a function  $f : [a, b] \rightarrow \mathbb{R}$  such that, for every  $\varepsilon > 0$ ,  $f$  is integrable on  $[a + \varepsilon, b]$  and  $\lim_{\varepsilon \rightarrow 0} \int_{a+\varepsilon}^b f(x) \, dx$  exists. Then we define the improper integral

$$\int_a^b f(x) \, dx \text{ to be } \lim_{\varepsilon \rightarrow 0} \int_{a+\varepsilon}^b f(x) \, dx.$$

even if the Riemann integral does not exist.

We can do similarly for  $[a, b - \varepsilon]$ , or integral to infinity:

$$\int_a^\infty f(x) \, dx = \lim_{b \rightarrow \infty} \int_a^b f(x) \, dx.$$

when it exists.