Adams spectral sequence of $\mathrm{tmf} \wedge \mathbb{RP}^\infty$

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The goal of this note is to outline the computation of the Adams spectral sequence of $\mathrm{tmf} \wedge \mathbb{RP}^\infty$. Essentially all differentials follow from the Leibniz rule, and products can be computed with a computer. The only work to be done is to organize the computation in order to conclude that we have indeed computed all differentials.

To do so, we need a complete calculation of the Adams E_2 page, which was done by Davis and Mahowald [1] (in their notation, $\Sigma^\infty \mathbb{RP}^\infty = P_1$). As usual, we have

$$\mathrm{Ext}_A^{s,t}(k, H_*(\mathrm{tmf} \wedge \Sigma^\infty \mathbb{RP}^\infty)) = \mathrm{Ext}_{A(2)}^{s,t}(k, H_*(\Sigma^\infty \mathbb{RP}^\infty)).$$

This group is free over v_2^8, where $|v_2^8| = (48, 8)$. Thus, to understand this group, it suffices to describe the generators under v_2^8. In the Davis–Mahowald description, these generators fall into 4 groups, and we colour-coded these in our chart in Figure 1. We shall go through the different groups in the coming sections, giving a formal description and describe the differentials that pertain to these groups. The differentials up to degree 96 are depicted in Figures 4 to 7. The range 96–192 is fairly similar and is depicted in Figure 8. Finally, v_2^{32} is permanent and so all differentials are v_2^{32}-periodic.

Conventions

We set $k = \mathbb{F}_2$, and write $x_{t-s,s}$ for a generator in the corresponding bidegree.

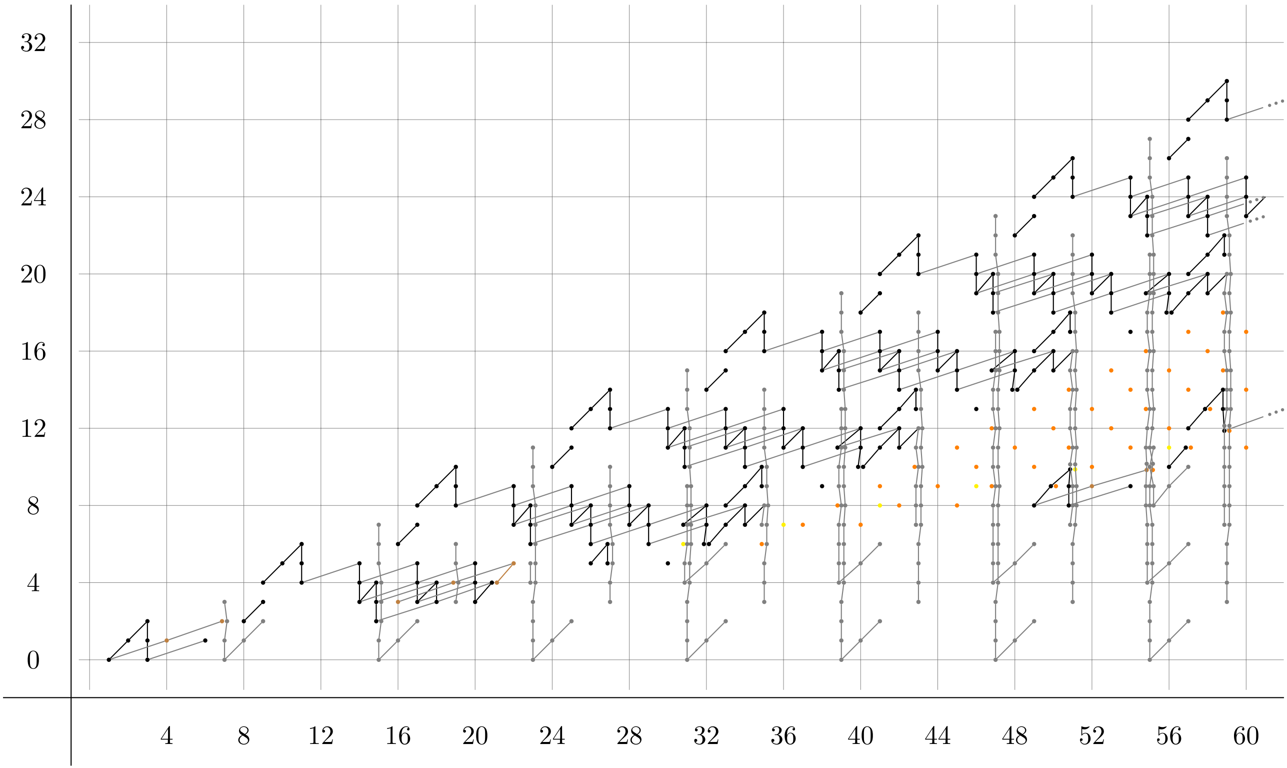


Figure 1: $\mathrm{Ext}_{A(2)}(k, H_*(P_1))$

1 ko TYPE CLASSES

We first deal with the gray classes that look like quotients of kos. To understand these classes, we use the cofiber sequence

$$\mathrm{tmf}^{hC_2} \to \mathrm{tmf}^{tC_2} \to \Sigma\mathrm{tmf}_{hC_2}$$

which induces a short exact sequence on homology, if we think of tmf^{hC_2} and tmf^{tC_2} as pro-spectra in the usual way. Moreover, by [2, Lemma 1.3], we know that

$$\mathrm{Ext}_c^{s,t}(k, H_*(\mathrm{tmf}^{tC_2})) \cong \bigoplus_{k\in\mathbb{Z}} \mathrm{Ext}_{A(1)}^{s,t}(k, k[8k]).$$

So the Ext groups of tmf^{tC_2} look like a bunch of ko's, and for degree reasons, its Adams spectral sequence must degenerate.

We claim that the gray classes are in the image of $\mathrm{Ext}_c^{s,t}(k, H_*\mathrm{tmf}^{tC_2})$, hence must be permanent. It suffices to prove that the generators under h_0 and v_1^4 are in the image, i.e. the classes in bidegree $(8k-1, 0)$. To do so, we note that they cannot be in the image of

$$\mathrm{Ext}_c^{s,t}(k, H_*(\mathrm{tmf}^{hC_2})) \to \mathrm{Ext}_c^{s,t}(k, H_*(\mathrm{tmf}^{tC_2})).$$

Indeed, the left-hand side is

$$\varprojlim \mathrm{Ext}_{A(2)}^{s,t}(k, H_*(D\Sigma_+^\infty \mathbb{RP}^n)).$$

The top dimensional cell in $D\Sigma_+^\infty\mathbb{RP}^n$ is always in degree 0. So the bigraded group $\mathrm{Ext}_A^{s,t}(k, H_*(\mathrm{tmf}^{\Sigma_+^\infty\mathbb{RP}^\infty}))$ has a bottom vanishing line equal to that of $\mathrm{Ext}_{A(2)}^{s,t}(k,k)$. In particular, the corresponding generators at $(8k-1, 0)$ are all below this line, so are mapped injectively into $\mathrm{Ext}_{A(2)}(k, H_*(P_1))$.

We now give a formal description of these classes. For any $i \in \mathbb{Z}$, we let $\mathrm{C}(i)$ be the chart of $\Sigma^{8i-1}\mathrm{ko}$ truncated to below the line $y = -\frac{x}{4} + 6i - 1$. Then the gray classes are given by $\bigoplus_{i\geq 1}\mathrm{C}(i)$ plus all its v_2^8-multiples. We depict $\mathrm{C}(3)$ in Figure 2 for reference.

2 v_2-PERIODIC CLASSES

We next look at the five brown classes. There is not much interesting to say about them. They have no periodicity apart from v_2^8. For degree reasons, only $x_{16,3}$ can potentially hit something, but the target is v_1 periodic while this is not. Note however that v_2^8 multiples of these need not be permanent, and indeed they will not be.

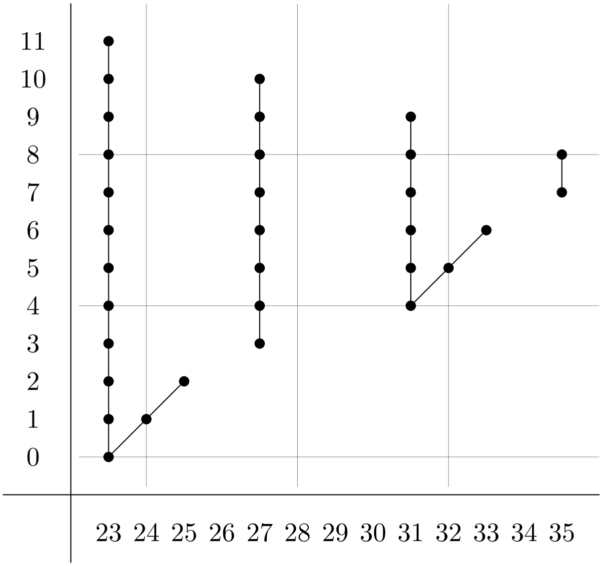


Figure 2: Depiction of C(3)

3 w-PERIODIC CLASSES

Next, we look at the orange and yellow classes. The yellow classes is a free $k[w]$-module on a single generator $x_{31,6}$, where $|w| = (5, 1)$. There is no actual element w that you can multiply with; rather, it is given by the Massey product $\langle h_1, h_2, -\rangle$.

However, some multiples of w are realized by elements in $\pi_*\mathrm{tmf}$, namely

$$\beta = w^3$$
$$g = w^4$$
$$\gamma = w^5$$

So w^k exists for $k \geq 3$ and is permanent for sufficiently large k. These yellow classes are in fact all permanent.

It is useful to note that $x_{31,6}$ is in fact itself w divisible, with $w^6 x_{1,0} = x_{31,6}$. Note that $w^4 x_{1,0} = g x_{1,0}$ is the sum of the two basis elements in bidegree $(21, 4)$. The two basis elements can be uniquely identified as follows — the brown class is the unique non-zero class that is v_1^4 torsion, while the black class is the unique non-zero class that is h_1 divisible.

These classes stay along for quite a while. The differentials that eventually kill them come from the differential

$$d_3(v_2^{16}) = w^{19}$$

in the Adams spectral sequence for tmf.

The orange classes form a free $k[w, v_1]$-module on a single generator $x_{35,6}$. Again v_1^4 exists and is given by, well, v_1^4. It is also convenient to note that

$$d_0 = w^2 v_1^2$$
$$e_0 = w^3 v_1$$
$$\alpha = w^2 v_1$$

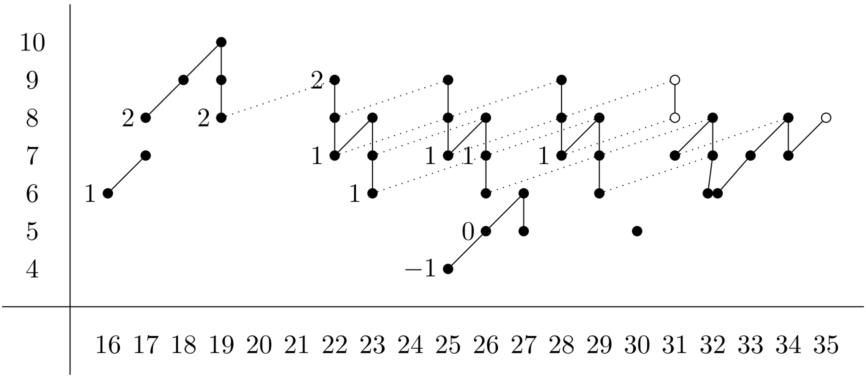
These have a fairly complicated differential pattern, but these all follow from the differentials in the Adams spectral sequence for tmf and the Leibniz rule. It is prudent to note that the Adams spectral sequence for tmf has

$$d_2(v_2^8) = g\alpha\beta = v_1 w^9,$$

so $w^{9+k} v_1^{1+j} x_{35,6} = 0$ for all $j, k \geq 0$ on the E_3 page, and we are left with a sequence $w^{9+k} x_{35,6}$ of dots separated by $(5, 1)$. In fact the sequence starts at $x_{20,3}$ with $w^2 x_{20,3} = x_{30,5}$ and $w^3 x_{20,3} = x_{35,6}$. These again eventually get killed by d_3's in a manner exactly analogous to the $x_{31,6}$'s.

4 v_1-PERIODIC CLASSES

We finally get to the black classes, which are v_1^4-periodic. A "unit" of this v_1 periodicity looks like this:

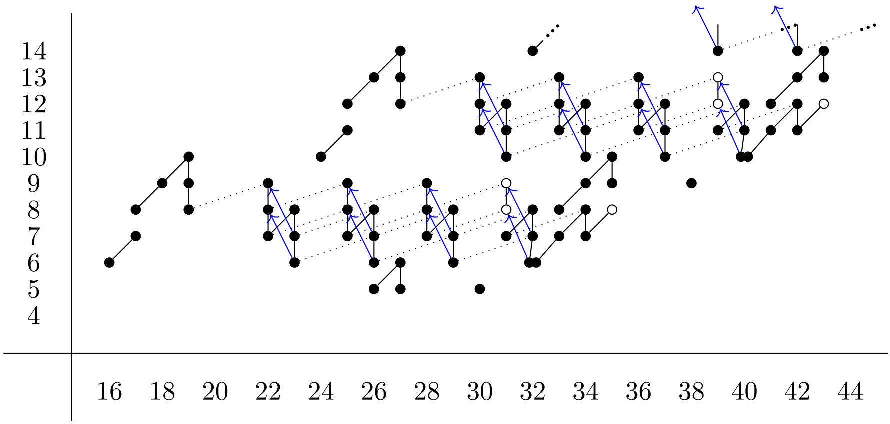


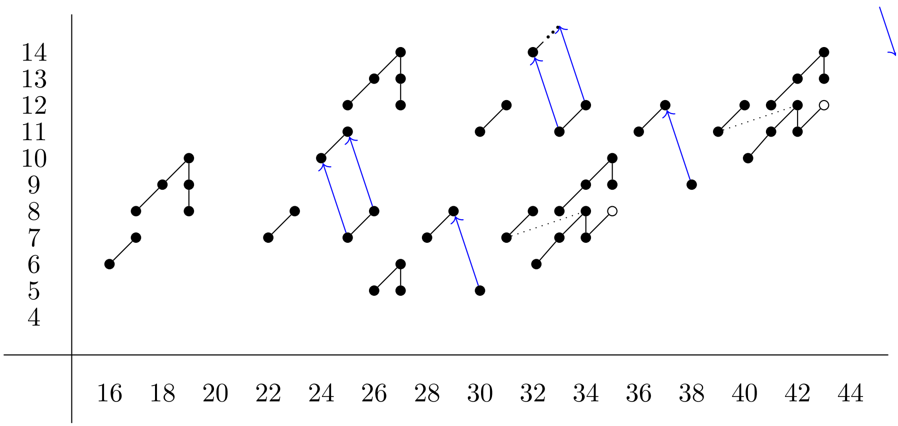
The numbers on the class denote how many times it is v_1^4 divisible, relative to the coordinates in the diagram. For example, $x_{17,8}$ can be divided by v_1^4 twice to give $x_{1,0}$, while $x_{25,4}$ doesn't even exist; only $v_1^4 x_{25,4}$ does.

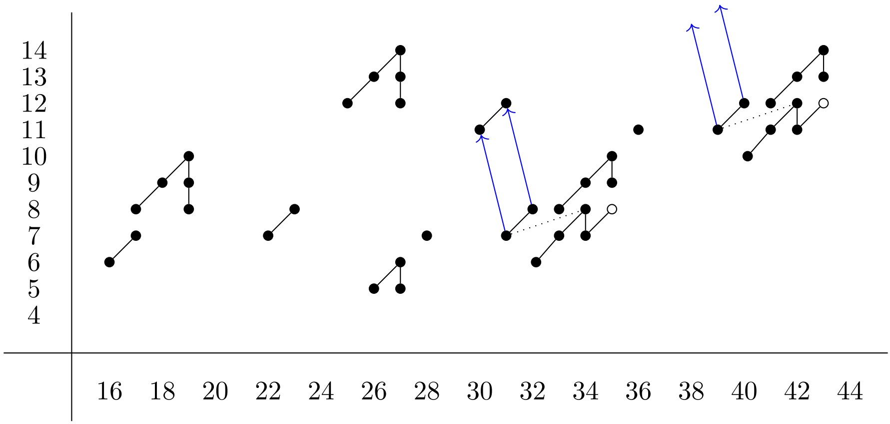
The divisibilities of unlabelled classes are determined by h_0 and h_1 products (if x divides, then so do h_0x and h_1x). If a class is completely unlabelled, then its label should be interpreted to be 0.

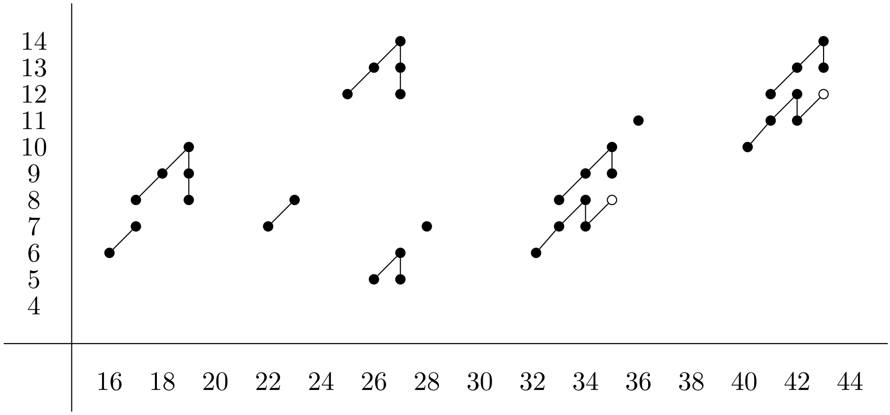
Finally, the hollow classes are not actually in the diagram, but come from the C's. Their role is merely to indicate multiplications.

The differentials in D follow from the differentials for tmf via the Leibniz rule again. They look as follows:









The two "hooks" with lower left corner at $(17, 8)$ and $(25, 4)$ are v_1^4 periodic. The classes left (including the hollow ones) are killed by elements in $k[v_1, w] \cdot x_{35,6}$.

5 Stems 97–192

The situation in stems 97–192 is very similar to the first 96 stems. In the Adams spectral sequence for tmf, we have

$$d_3(v_2^{16}) = w^{19}.$$

So in particular, all d_2's in this range are the same as in the first 96. Moreover, by the Leibniz rule, for any x, we have

$$d_3(xv_2^{16}) = d_3(x)v_2^{16} + w^{19}x.$$

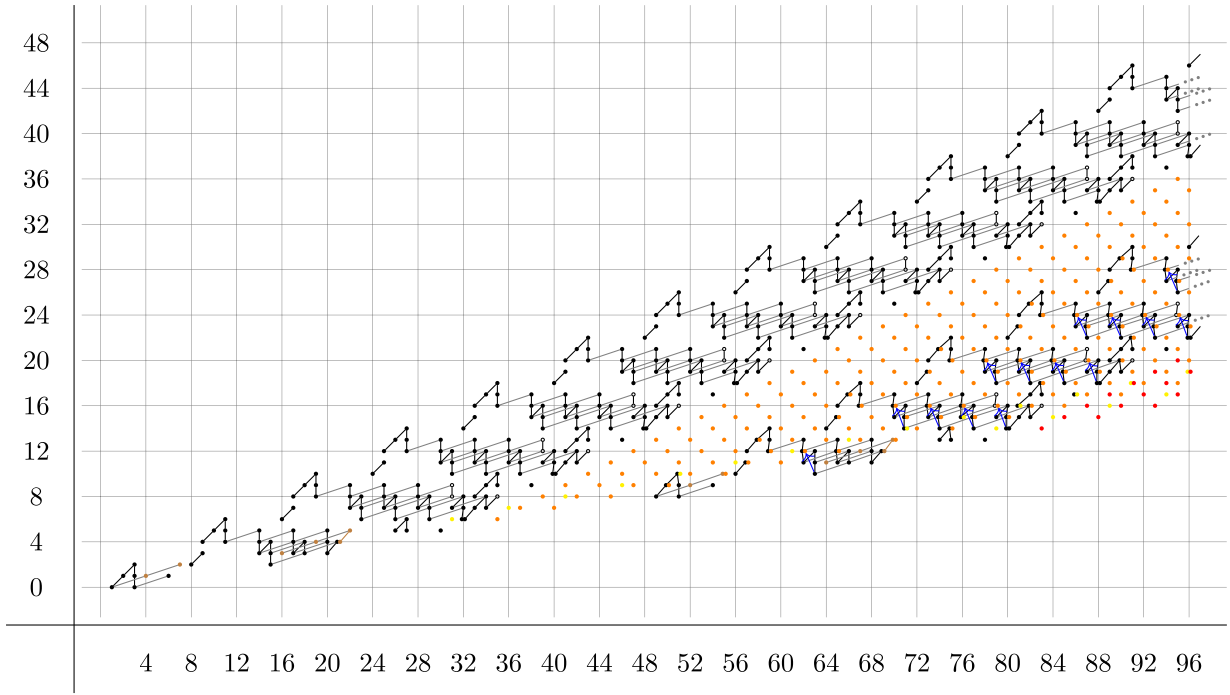


Figure 3: E_2 page for Adams spectral sequence of $\mathrm{tmf} \wedge \Sigma^\infty \mathbb{RP}^\infty$ without differentials

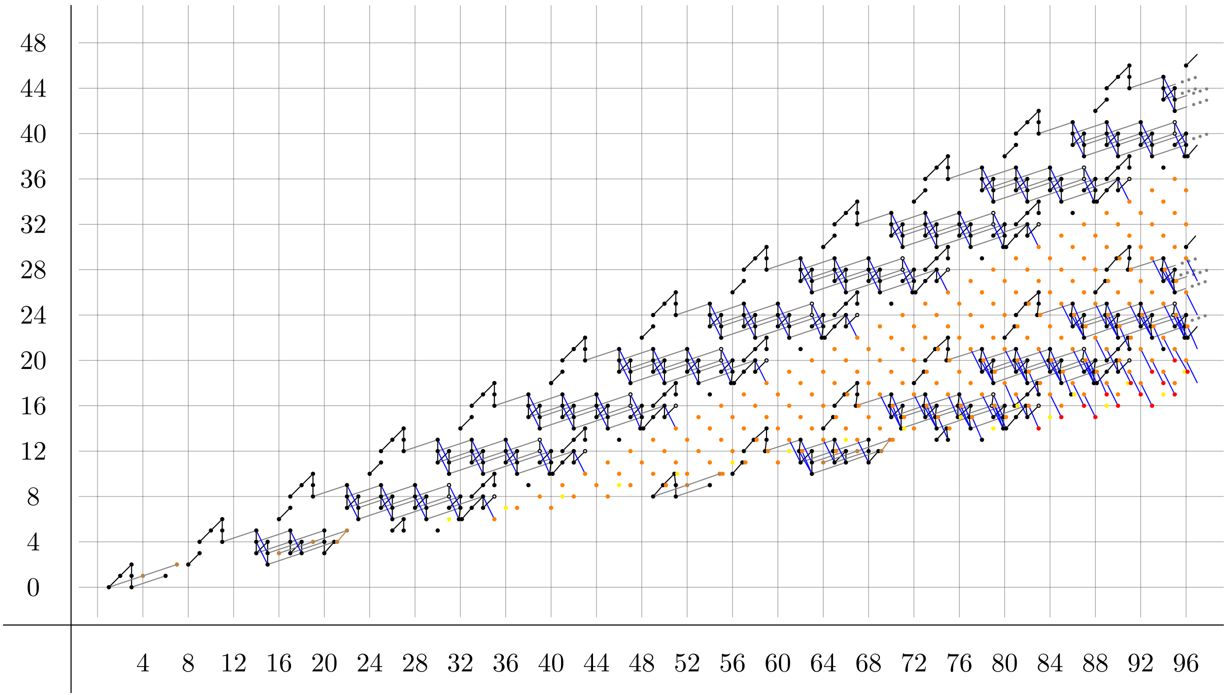


Figure 4: E_2 page for Adams spectral sequence of $\mathrm{tmf} \wedge \Sigma^\infty \mathbb{RP}^\infty$

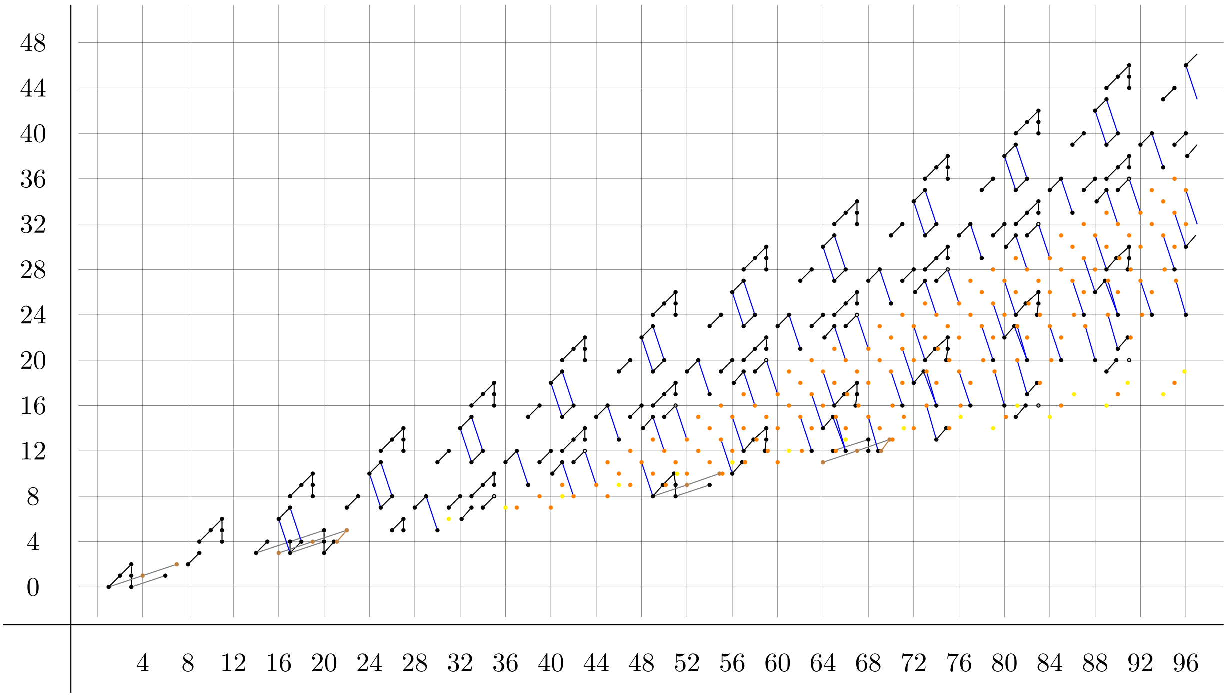


Figure 5: E_3 page for Adams spectral sequence of $\mathrm{tmf} \wedge \Sigma^\infty \mathbb{RP}^\infty$

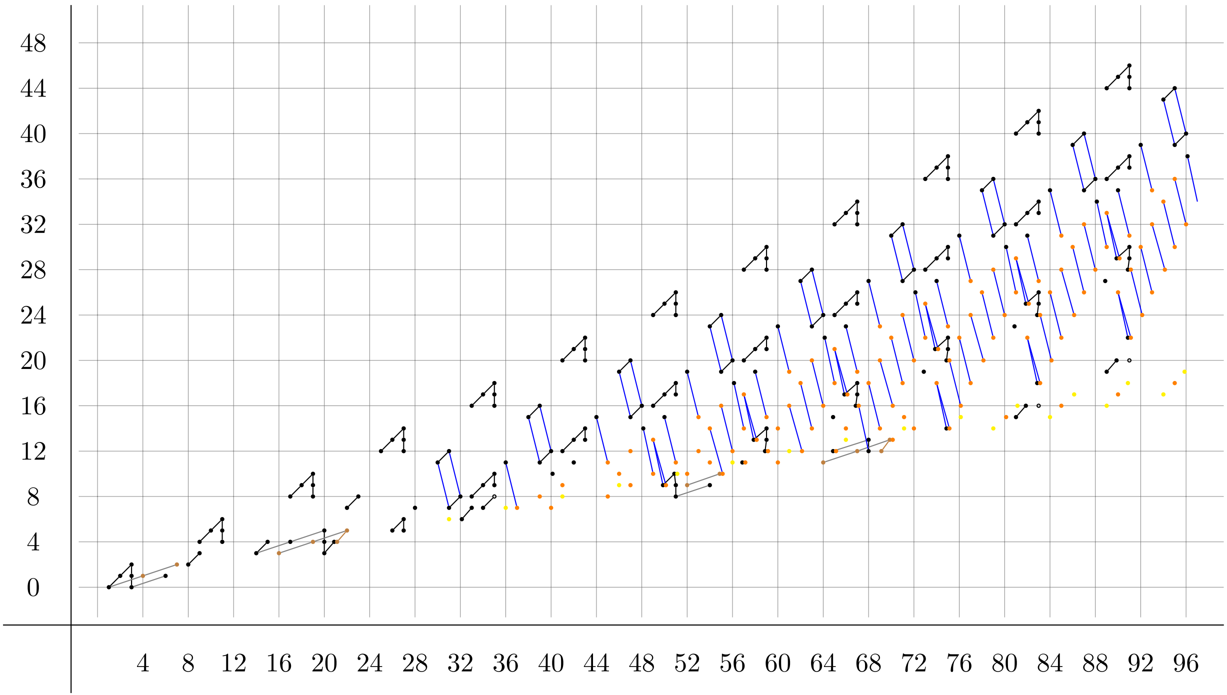


Figure 6: E_4 page for Adams spectral sequence of $\mathrm{tmf} \wedge \Sigma^\infty \mathbb{RP}^\infty$

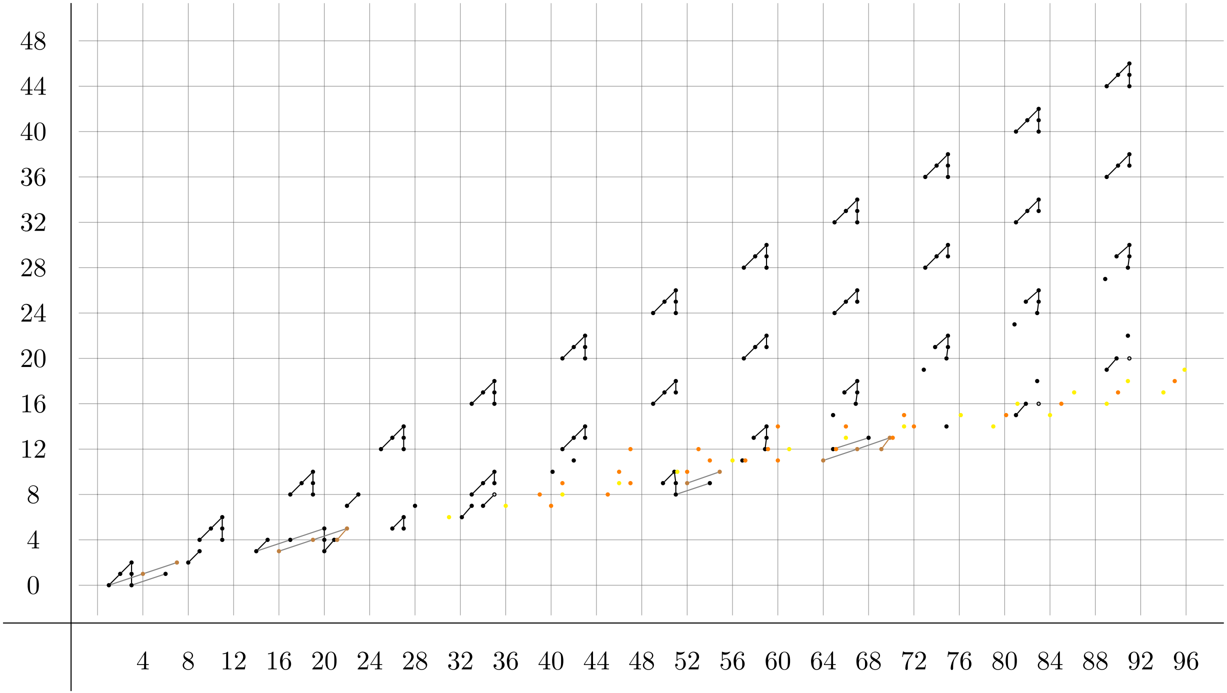


Figure 7: E_∞ page for Adams spectral sequence of tmf $\wedge\, \Sigma^\infty \mathbb{RP}^\infty$

For most of the terms, the w multiples have already been killed by d_2's, so the d_3's get preserved. The extra d_3's we get come from w multiples of $v_2^{16}x_{1,0}$, $v_2^{24}x_{1,0}$ and $v_2^{16}x_{20,3}$.

The d_4's are also preserved by v_2^{16}, except for the d_4 on $w^4x_{35,6}$, which supports a d_3 instead. This follows from the fact that our d_4's are v_1^4 periodic and $v_1^4v_2^{16}$ is permanent.

We depict the interesting d_3's in Figure 8, omitting the classes that get killed by differentials propagated from the first 96 stems. This is included because one has to do a bit of book keeping to keep track of which of the w^{19+k} multiples actually get killed for small k.

References

[1] Donald M. Davis and Mark Mahowald. Ext over the subalgebra A_2 of the Steenrod algebra for stunted projective spaces. In *Current trends in algebraic topology, Part 1 (London, Ont., 1981)*, volume 2 of *CMS Conf. Proc.*, pages 297–342. Amer. Math. Soc., Providence, RI, 1982.

[2] W. H. Lin, D. M. Davis, M. E. Mahowald, and J. F. Adams. Calculation of Lin's Ext groups. *Math. Proc. Cambridge Philos. Soc.*, 87(3):459–469, 1980.

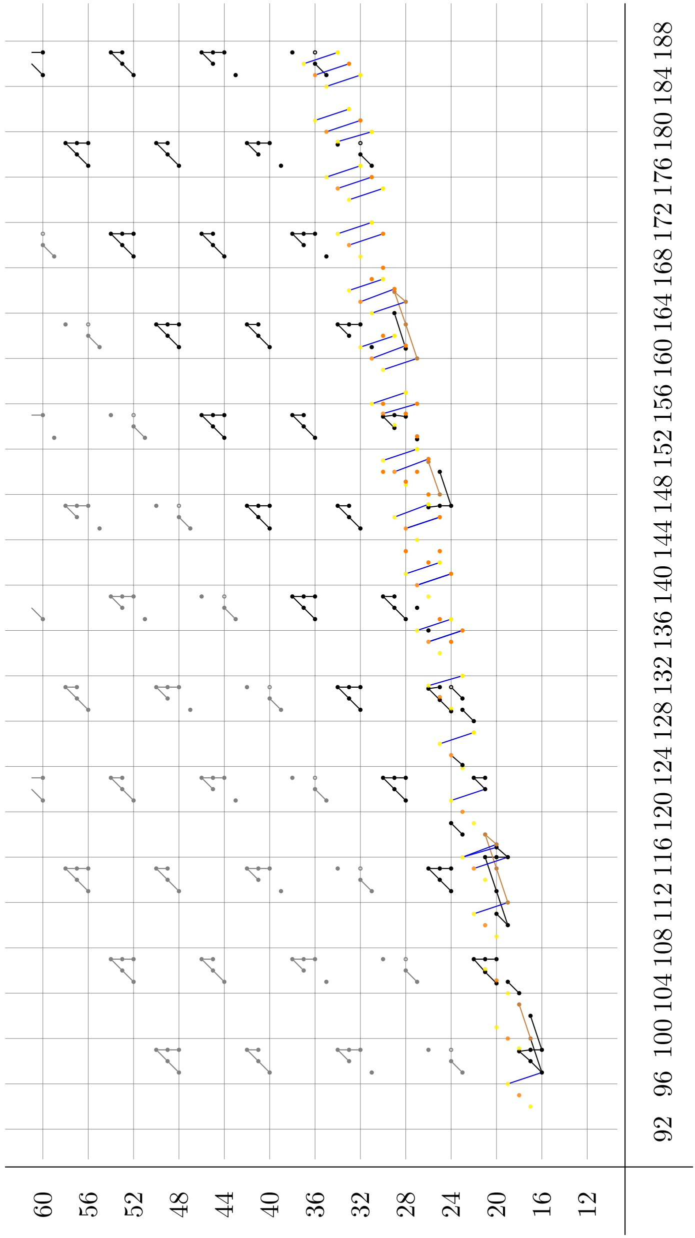


Figure 8: Differentials in 96–192

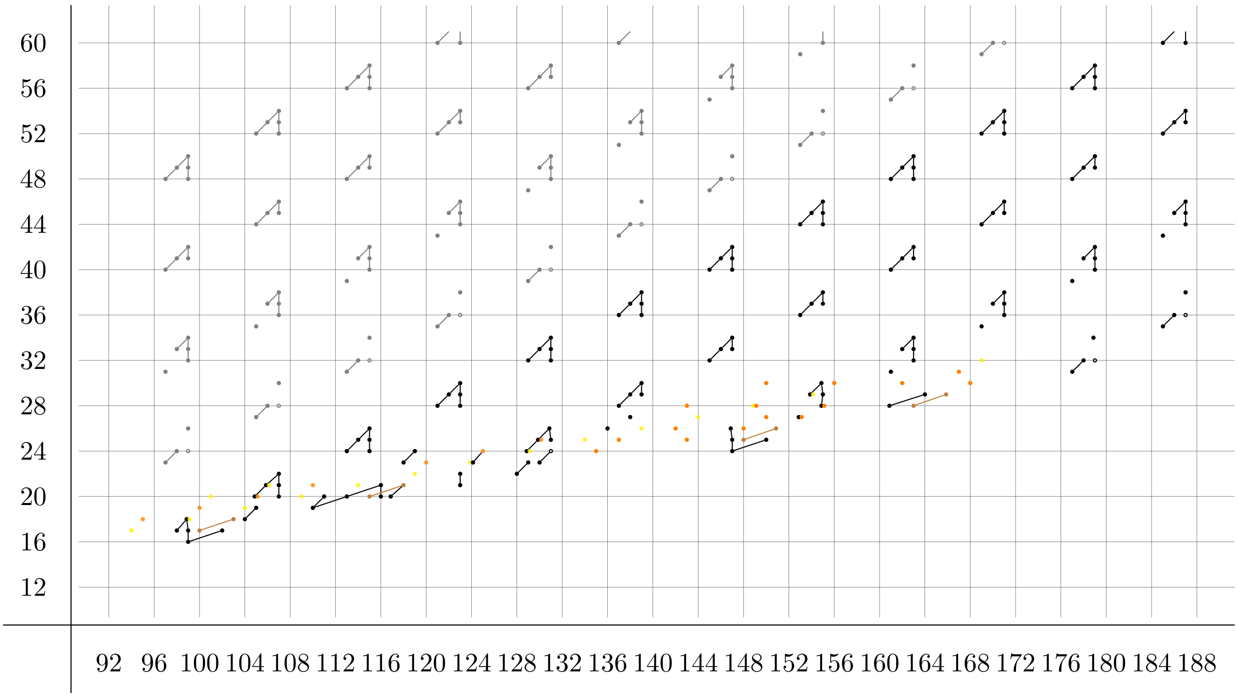


Figure 9: Permanent classes in 96–192