Construction of v_1 and v_2 self-maps

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In these notes, I will define the notion of a v_n self map, and prove their existence for $n = 1$ and 2. These maps are used to construct infinite families in the homotopy groups of spheres.

1 Motivation and definitions

The Adams–Novikov spectral sequence is a spectral sequence

$$\mathrm{Ext}^{s,t}_{BP_*BP}(BP_*, BP_*) \Rightarrow \pi_{t-s}(\mathbb{S}).$$

We will abbreviate $\mathrm{Ext}^{s,t}_{BP_*BP}(BP_*, M)$ as $\mathrm{Ext}^{s,t}(C)$, and sometimes omit the t.

To use this to construct elements in $\pi_{t-s}(\mathbb{S})$, we have to do three things:

1. Find an element in $\mathrm{Ext}^{s,t}(BP_*)$
2. Show that it doesn't get hit by differentials
3. Show that all differentials vanish on it.

All three steps are difficult, except with some caveats.

1. We can do this if $s = 0$.
2. We can do this if s is small enough.
3. We can do this if we know the map of spheres actually exists, and want to show it is non-zero.

Of these three caveats, (1) is perhaps the worst, because the $s = 0$ line is boring. To make better use of our ability to calculate Ext^0, suppose we have a short exact sequence of comodules, such as

$$0 \longrightarrow BP_* \xrightarrow{p} BP_* \longrightarrow BP_*/p \longrightarrow 0.$$

We then get a coboundary map

$$\delta : \mathrm{Ext}^0(BP_*/p) \to \mathrm{Ext}^1(BP_*).$$

So we can use this to produce elements in $\mathrm{Ext}^1(BP_*)$. To understand the geometry of this operation, so that we can do (3), we use the following lemma, whose proof is a fun diagram chase:

Lemma. *Suppose $A \to B \to C \to \Sigma A$ is a cofiber sequence, and suppose that the map $BP_*A \to BP_*B$ is injective, so that we have a short exact sequence*

$$0 \to BP_*A \to BP_*B \to BP_*C \to 0.$$

Suppose $f : \mathbb{S}^? \to C$ is a map, whose corresponding element in the Adams spectral sequence is $\hat{f} \in \mathrm{Ext}^s(BP_*C)$. Then the composition $\mathbb{S}^? \to C \to \Sigma A$ corresponds to $\delta\hat{f} \in \mathrm{Ext}^{s+1}(BP_*C)$. In particular, $\delta\hat{f}$ is a permanent cycle. □

The slogan is

If δ comes from geometry, it sends permanent cycles to permanent cycles.

For the short exact sequence above, we can realize it as the BP homology of

$$\mathbb{S} \xrightarrow{p} \mathbb{S} \longrightarrow \mathbb{S}/p \equiv V(0).$$

Recall that $\mathrm{Ext}^0(BP_*/p) = \mathbb{F}_p[v_1]$. If we can find a map $\tilde{v}_1 : \mathbb{S}^{2p-2} \to V(0)$ that gives $v_1 \in \mathrm{Ext}^0(BP_*/p)$, then we know $\delta(v_1) \in \mathrm{Ext}^1(BP_*/p)$ is a permanent cycle. Since this has $s = 1$, no differentials can hit it, and as long as $\delta(v_1) \neq 0 \in \mathrm{Ext}^1(BP_*)$, which is a *purely algebraic problem*, we get a non-trivial element in the homotopy groups of sphere.

This is actually not a very useful operation to perform, because the way we are going to construct $\tilde{v}_1$ is by analyzing the Adams–Novikov spectral sequence for $V(0)$, which is not very much easier than finding the element in $\mathrm{Ext}^1(BP_*)$ directly.

But if we can promote this to a map $v_1 : \Sigma^{2p-2}V(0) \to V(0)$ that induces multiplication by v_1 on BP_*, then we can form the composition

$$\mathbb{S}^{t(2p-2)} \hookrightarrow \Sigma^{t(2p-2)}V(0) \xrightarrow{v_1^t} V(0)$$

which represents $v_1^t \in \mathrm{Ext}^0(BP_*/p)$, where the first map is the canonical quotient map $\mathbb{S} \to \mathbb{S}/p = V(0)$. We then know that $\delta(v_1^t) \in \mathrm{Ext}^1(BP_*)$ is permanent, and the above argument goes through. Thus, by constructing a single map $v_1 : \Sigma^{2p-2}V(0) \to V(0)$, we have found an *infinite* family of permanent cycles in Ext^1, knowing by magic that all the differentials from it must vanish. It is in fact true that for $p > 2$, the map v_1 exists and they are all non-trivial. These elements are known as α_t.

The map v_1 is known as a v_1 *self map* of $V(0)$. If we are equipped with such a map, we can play the same game with the cofiber sequence

$$\Sigma^{2p-2}V(0) \xrightarrow{v_1} V(0) \to V(1).$$

We then know that $BP_*V(1) = BP_*/(p, v_1)$, and we have a short exact sequence

$$0 \longrightarrow BP_*/p \xrightarrow{v_1} BP_*/p \longrightarrow BP_*/(p, v_1) \longrightarrow 0.$$

Again we know that $\mathrm{Ext}^0(BP_*/(p, v_1)) = \mathbb{F}_p[v_2]$, and we can seek a v_2 *self map* $v_2 : \Sigma^{2p^2-2}V(1) \to V(1)$ that induces multiplication by v_2 on BP_*. If we can do so, then we know that $\delta(v_2^t) \in \mathrm{Ext}^1(BP_*/p)$ is a permanent cycle, and hence $\delta\delta(v_2^t) \in \mathrm{Ext}^2(BP_*)$ is also a permanent cycle. This gives us a second sequence of elements in the stable homotopy group of spheres. Moreover, in this case the non-triviality is again an *algebraic* problem of showing that $\delta\delta(v_2^t) \neq 0 \in \mathrm{Ext}^2(BP_*)$, since no differentials can hit it. These elements are known as β_t.

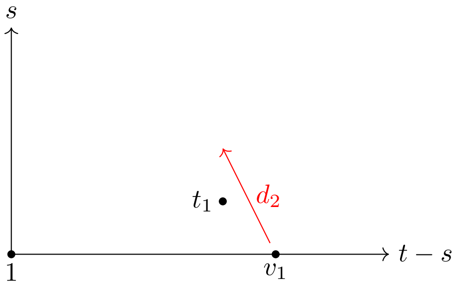
In these notes, I will construct the v_1 self maps for $p > 2$ and v_2 self maps for $p > 3$. It is true that the corresponding α_t and β_t are in fact non-zero, but I will not prove it here. These maps were first constructed by Adams and Smith (for v_1 and v_2 respectively), but they had to do more work because they didn't have BP and the Adams–Novikov spectral sequence.

We can of course continue this process to seek v_n self maps for larger n, and you should be glad to hear that this will become prohibitively difficult way before we run out of Greek letters.

2 Construction of v_1 self maps

We wish to construct a map $\Sigma^{2p-2}V(0) \to V(0)$ inducing multiplication by v_1 on BP_* homology. The strategy is to construct a map $\mathbb{S}^{2p-2} \to V(0)$ that induces multiplication by v_1 on BP_* homology, and then extend it to a map $\Sigma^{2p-2}V(0) \to V(0)$ by obstruction theory.

First consider the BP Adams–Novikov spectral sequence for $V(0)$. In degrees up to $2p-2$, the spectral sequence looks like



where $v_1 \in (2p-2, 0)$ and $t_1 \in (2p-3, 1)$. If $p=2$, then we have an extra t_1^2 which will be right above v_1.

In either case, we see that there is no room for extra differentials. So we see that

Lemma. *There is a map $\tilde{v}_1 : \mathbb{S}^{2p-2} \to V(0)$ that induces multiplication by v_1 on BP_*. If $p > 2$, then this map has order p and is unique.* □

Since $V(0) = \mathbb{S}/p$, the map $\tilde{v}_1$ having order p is the same as it extending to a map $\Sigma^{2p-2} V(0)$. Thus, we deduce that

Theorem. *If $p > 2$, then there is a map $v_1 : \Sigma^{2p-2} V(0) \to V(0)$ that induces multiplication by v_1 on BP_*.*

In the case $p = 2$, we know $\pi_2 V(0) = \mathbb{Z}/4\mathbb{Z}$ or $\mathbb{Z}/2 \oplus \mathbb{Z}/2$. If it is $\mathbb{Z}/4\mathbb{Z}$, then this map has order 4 and does not lift to a map $\Sigma^2 V(0) \to V(0)$. This is indeed the case, as one can check using the $H\mathbb{F}_2$ Adams spectral sequence, so we do not have a v_1 self map at $p = 2$.

3 Construction of v_2 self maps

We next attempt to construct v_2 self maps. It turns out there is no v_2 self map when $p = 3$, so in this section we will exclusively concentrate on the case $p > 5$.

We first sketch the argument to see how far in the Adams–Novikov spectral sequence we have to go. There is an element $v_2 \in \mathrm{Ext}^{0, 2p^2-2}(BP_*, BP_*/(p, v_1))$. We want to show this survives to give a map $\mathbb{S}^{2p^2-2} \to V(1)$, and so we will have to understand the $t - s = 2p^2 - 3$ column, which we will find to be empty.

If we further find that this map has order p, then it factors through $\Sigma^{2p^2-2} V(0)$. This is the same as saying there are no elements in the $t - s = 2p^2 - 2$ column apart from (multiples of) v_2.

Finally, to show that this descends to a map from $\Sigma^{2p^2-2} V(1)$, we precompose with the (suspension of the) v_1 self map of $V(0)$ to get a map

$$\Sigma^{2p^2+2p-4} V(0) \xrightarrow{\Sigma^{2p^2-2} v_1} \Sigma^{2p^2-2} V(0) \longrightarrow V(1),$$

and we have to show that this vanishes. We shall show that $[\Sigma^{2p^2+2p-4}V(0), V(1)] = 0$ using the long exact sequence

$$[\mathbb{S}^{2p^2+2p-3}, V(1)] \to [\Sigma^{2p^2+2p-4}V(0), V(1)] \to [\mathbb{S}^{2p^2+2p-4}, V(1)]$$

given by the cofiber sequence $\mathbb{S} \xrightarrow{p} \mathbb{S} \to V(0)$.

So to show that v_2 exists, we have to prove the following:

Theorem. *Suppose $p > 3$. Then $\mathrm{Ext}^{s,t}(BP_*/(p, v_1)) = 0$ for*

$$t - s = 2p^2 - 3, \quad 2p^2 + 2p - 3, \quad 2p^2 + 2p - 4.$$

Moreover, the column $t - s = 2p^2 - 2$ is generated by v_2 of order p.

So we will have to compute the Adams–Novikov spectral sequence up to $t - s \leq 2p^2 + 2p - 3$. In Ravenel's green book, the computation was done up to $\sim p^3$, but for our range, we can get away with doing some simple counting.

In this range, the generators in BP_*BP that show up in the cobar complex are t_1, t_2 and v_2. Note that there are two ways we can multiply t_1 — either in BP_*BP itself, or as $t_1 \otimes t_1$ in the cobar complex. In either case, any appearance of t_1 will contribute at least $2p-3$ to $t-s$. Similarly, t_2 and v_2 contribute $2p^2-3$ and $2p^2-2$ respectively. The assumption that $p \geq 5$ allows us to perform the following difficult computation:

Theorem. *If $p \geq 5$, then $2p - 3 \geq 7$.* $\square$

Thus, we can enumerate all the terms that appear in the cobar complex in the range $t - s \leq 2p^2 + 2p - 3$:

Element	$t - s$
Terms involving only t_1	??
v_2	$2p^2 - 2$
t_2	$2p^2 - 3$
$t_1 v_2$	$2p^2 + 2p - 5$
$t_1 t_2$	$2p^2 + 2p - 5$
$t_1 \otimes t_2$	$2p^2 + 2p - 6$
$t_2 \otimes t_1$	$2p^2 + 2p - 6$

The term t_2 is in a problematic column, but it shall not concern us for two reasons. Firstly, it has $s = 1$, so it wouldn't get hit by our differentials. Secondly, we can calculate that $\mathrm{d}(t_2) = t_1 \otimes t_1^p$ (e.g. see 4.3.15 of Ravenel's Green book), and so t_2 does not actually appear in the Adams spectral sequence. So we would be done if we can show that there are no purely t_1 terms appearing in the four columns of the theorem.

The element t_1 is a primitive element, and the following lemma is convenient:

Lemma. *Let $\Gamma = P(x)$ be a Hopf algebra over $\mathbb{F}_p$ $(p > 2)$ on one primitive generator in even degree. Then*

$$\mathrm{Ext}_\Gamma(\mathbb{F}_p, \mathbb{F}_p) = E(h_i : i = 0, 1, \ldots) \otimes P(b_i : i = 0, 1, \ldots)$$

where

$$h_i = x^{p^i} \in \mathrm{Ext}^1, \quad b_i = \sum_{0<j<p} \frac{1}{p}\binom{p}{j} x^{jp^i} \otimes x^{(p-j)p^i} \in \mathrm{Ext}^2\,. \qquad \square$$

The presence of elements not involving t_1 will not increase the number of purely t_1 cohomology classes, but may kill off some if the coboundary of some term is purely t_1 (e.g. $\mathrm{d}(t_2)$). So it is enough to see that there are no product of the h_i and b_i fall into the relevant columns.

In degrees $t - s \leq 2p^2 + 2p - 4$, we have generators

$$1 \in \mathrm{Ext}^{0,0}, \quad h_0 \in \mathrm{Ext}^{1,2p-2}, \quad h_1 \in \mathrm{Ext}^{1,2p^2-2p}, \quad b_0 \in \mathrm{Ext}^{2,2p^2-2p},$$

We see that no product of these can enter the column we care about. So we are done.

As a side note, in the case $p = 3$, we have $2p^2 + 2p - 3 = 21$, and $h_1 b_0 \in \mathrm{Ext}^{3,24}$ is a non-trivial element with $t - s = 2p^2 + 2p - 3$.