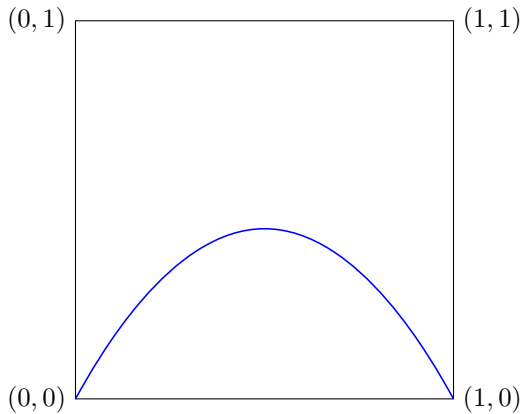


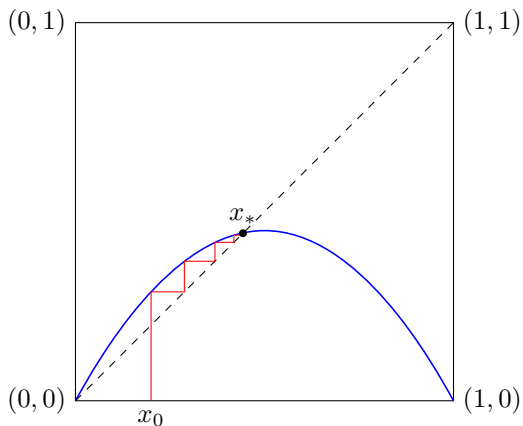
Period 3 orbits

Dexter Chua

Suppose we are given a continuous function $f : [0, 1] \rightarrow [0, 1]$. For example, the graph of f may look like this:

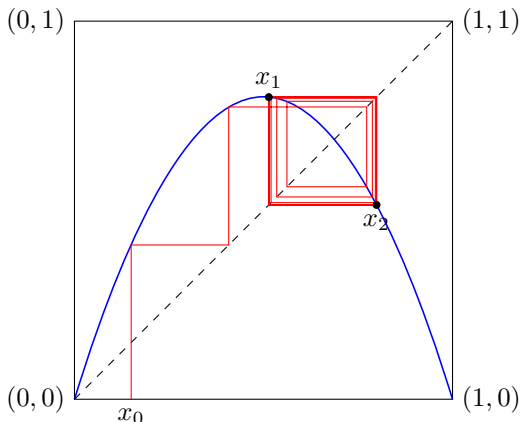


One interesting thing to do is to randomly pick a point $x \in [0, 1]$, keep applying f , and see where we end up. We can visualize this process via a cobweb diagram, drawn as follows:



We see that we converge towards the point x_* . In particular, if we start at x_* , we stay there all the time — x_* is a *fixed point*.

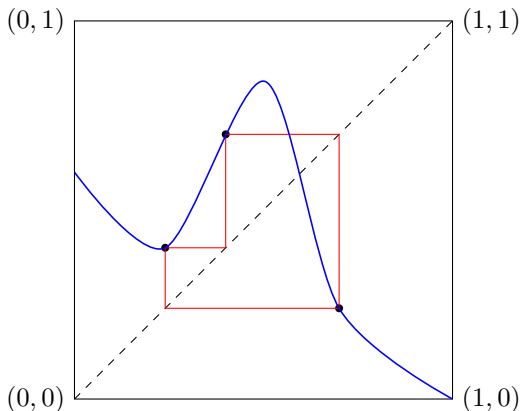
Some choices of f give more complicated behaviour. For example, in the following diagram



we see that in the limit, the point oscillates between x_1 and x_2 . In this case, x_1 and x_2 are 2-periodic points. Perhaps if we want to care about the limiting behaviour, we should understand these periodic points.

Definition. Let $f : [0, 1] \rightarrow [0, 1]$ be a function. A point $x \in [0, 1]$ is *n-periodic* if $f^n(x) = x$ and $f^k(x) \neq x$ for $0 < k < n$.

For example, for the following choice of f , we have a 3-periodic point:



We are now ready to state the desired theorem.

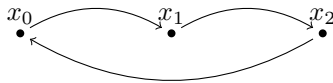
Theorem. Let $f : [0, 1] \rightarrow [0, 1]$ be a continuous function. If f has a 3-periodic point, then f has an n -periodic point for all n .

Let's try to understand this. A good start might be to find a fixed point, i.e. a 1-periodic point. We can achieve this using the intermediate value theorem. From now on, we fix a single continuous function f .

Lemma. Let $I \subseteq [0, 1]$ be a (closed) subinterval. If $f(I) \supseteq I$, then I contains a fixed point.

Proof. Write $I = [a, b]$. Then since $f(I) \supseteq I$, we in particular have $a, b \in f(I)$. So we can find $a', b' \in I = [a, b]$ such that $f(a') = a$ and $f(b') = b$. Then we have $f(a') - a' = a - a' \leq 0$ and $f(b') - b' = b - b' \geq 0$. Since $x \mapsto f(x) - x$ is continuous, we can find some c such that $f(c) - c = 0$, i.e. c is a fixed point. \square

So we just need to find an interval I such that $f(I) \supseteq I$. We suppose the three points in the 3-periodic orbit are $x_0 < x_1 < x_2$. We may wlog assume that $f(x_0) = x_1$, $f(x_1) = x_2$ and $f(x_2) = x_0$.



Then $f([x_1, x_2])$ is a closed subinterval containing x_0 and x_2 . So it in particular contains $[x_1, x_2]$. So we have found a fixed point.

How about periodic points of larger period? If we want to find an n -periodic point, we might try to look for fixed points of f^n . This is in some sense a good strategy, but it is not good enough. If we are given a fixed point f^n , we know it has period *at most* n . Perhaps it is secretly a fixed point of f . So we need to be a bit more delicate than that.

We first introduce a convenient notation:

Notation. If $I, J \subseteq [0, 1]$ are subintervals, we write $I \rightarrow J$ if $f(I) \supseteq J$.

We have seen that if $I \rightarrow I$, then I contains a fixed point of f . The following is the desired strengthening of the lemma.

Theorem. If $I_0 \rightarrow I_1 \rightarrow I_2 \rightarrow \dots \rightarrow I_{n-1} \rightarrow I_0$, then there exists an $x \in I_0$ such that $f^k(x) \in I_k$ for $k = 1, \dots, n-1$ and $f^n(x) = x$.

This is trivial if we don't require $f^k(x) \in I_k$, since $f^n(I) \supseteq I$, and we can apply the lemma. The (slightly) hard part is getting the points to lie in the I_k .

Proof. Note that if $I, J \subseteq [0, 1]$ are closed subintervals with $f(I) \supseteq J$, then $f^{-1}(J)$ is a disjoint union of closed intervals contained in I , each of which has image J . By picking one of these, we get a subinterval $K \subseteq I$ such that $f(K) = J$.

Now if $I_0 \rightarrow I_1 \rightarrow I_2 \rightarrow \cdots \rightarrow I_{n-1} \rightarrow I_0$, then we can find some $J_{n-1} \subseteq I_{n-1}$ such that $f(J_{n-1}) = I_0$. We still have $f(I_{n-2}) \supseteq J_{n-1}$, so we have a sequence

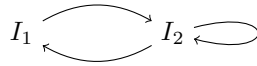
$$I_0 \rightarrow I_1 \rightarrow I_2 \rightarrow \cdots \rightarrow I_{n-2} \rightarrow J_{n-1} \rightarrow I_0.$$

We can now replace I_{n-2} with a J_{n-2} as above, and keep going on to obtain a new sequence

$$J_0 \rightarrow J_1 \rightarrow J_2 \rightarrow \cdots \rightarrow J_{n-2} \rightarrow J_{n-1} \rightarrow I_0.$$

such that $J_k \subseteq I_k$, and $f(J_k) = J_{k+1}$ for all k . Then since $f^n(J_0) = I_0 \supseteq J_0$, it follows that we can find some $x \in J_0$ such that $f^n(x) = x$. By assumption, $f^k(x) \in J_k \subseteq I_k$. So we are done. \square

We are essentially done with the proof. We set $I_1 = [x_0, x_1]$ and $I_2 = [x_1, x_2]$. Then we have $I_2 \rightarrow I_2$, $I_1 \rightarrow I_2$ and $I_2 \rightarrow I_1$. If we feel like drawing diagrams, we can draw it as



Now we have a sequence of the form

$$I_1 \rightarrow I_2 \rightarrow I_2 \rightarrow \cdots \rightarrow I_2 \rightarrow I_1$$

for any number of copies of I_2 . So for each n , we get some $x \in I_1$ such that $f^n(x) = x$, and crucially, $f^k(x) \in I_2$ for all $0 < k < n$. To see that x is actually of period n , we need to check that $f^k(x) \neq x$ for $0 < k < n$. To do so, we simply have to note that $I_1 \cap I_2 = \{x_1\}$. Thus if $f^k(x) = x$ for some k , then x must be x_1 . We can check manually that x_1 does not satisfy the required property, since $f^2(x_1) = x_0 \notin I_2$. This concludes the proof.

This result is a special case of a more general theorem called Sharkovskii's theorem. The statement is as follows:

Theorem. *Define an ordering on the naturals by*

$$\begin{array}{ccccccc} 3 & \triangleright & 5 & \triangleright & 7 & \triangleright & \cdots \triangleright \\ 2 \cdot 3 & \triangleright & 2 \cdot 5 & \triangleright & 2 \cdot 7 & \triangleright & \cdots \triangleright \\ 2^2 \cdot 3 & \triangleright & 2^2 \cdot 5 & \triangleright & 2^2 \cdot 7 & \triangleright & \cdots \triangleright \\ \vdots & & \vdots & & \vdots & & \cdots \triangleright \\ 2^3 & \triangleright & 2^2 & \triangleright & 2^1 & \triangleright & 1 \end{array}$$

Then if $f : [0, 1] \rightarrow [0, 1]$ has an n -periodic point and $n \triangleright m$, then there is an m -periodic point.

One way to prove this is to use adapt the above strategy with some clever choices of the intervals involved.

As a concluding remark, recall that we led ourselves into looking at periodic points when we tried to understand the limiting behaviour as we keep iterating f . In the first two examples, the fixed point and the 2-periodic points were *stable*, namely if we start somewhere near the periodic points and keep applying f , the point converges towards the fixed point or cycle. In general, there is no reason to believe that the periodic points given by our theorem will be stable. This makes it slightly less interesting — there is no way we can "naturally" discover these periodic points as we did at the beginning. We must have started at exactly the periodic point to discover the periodicity.