Motivic Homotopy Theory

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Throughout the talk, S is a quasi-compact and quasi-separated scheme.

Definition 0.1. Let Sm_S be the category of finitely presented smooth schemes over S.

Since we impose the finitely presented condition, this is an essentially small (1-)category.

The starting point of motivic homotopy theory is the ∞-category $\mathcal{P}(\mathrm{Sm}_S)$ of presheaves on Sm_S. This is a symmetric monoidal category under the Cartesian product. Eventually, we will need the pointed version $\mathcal{P}(\mathrm{Sm}_S)_*$, which can be defined either as the category of pointed objects in $\mathcal{P}(\mathrm{Sm}_S)$, or the category of presheaves with values in pointed spaces. This is symmetric monoidal ∞-category under the (pointwise) smash product.

In the first two chapters, we will construct the unstable motivic category, which fits in the bottom-right corner of the following diagram:

$$\begin{array}{ccc}
\mathcal{P}(\mathrm{Sm}_S) & \xrightarrow{L_{\mathrm{Nis}}} & L_{\mathrm{Nis}}\mathcal{P}(\mathrm{Sm}_S) \equiv \mathrm{Spc}_S \\
\big\downarrow \widetilde{L_{\mathbb{A}^1}} & & \big\downarrow L_{\mathbb{A}^1} \\
L_{\mathbb{A}^1}\mathcal{P}(\mathrm{Sm}_S) & \xrightarrow{\widetilde{L_{\mathrm{Nis}}}} & L_{A^1 \wedge \mathrm{Nis}}\mathcal{P}(\mathrm{Sm}_S) \equiv \mathrm{Spc}_S^{\mathbb{A}^1} \equiv \mathcal{H}(S)
\end{array}$$

The first chapter will discuss the horizontal arrows (i.e. Nisnevich localization), and the second will discuss the vertical ones (i.e. $\mathbb{A}^1$-localization).

Afterwards, we will start "doing homotopy theory". In chapter 3, we will discuss the motivic version of homotopy groups, and in chapter 4, we will discuss Thom spaces.

In chapter 5, we will introduce the stable motivic category, and finally, in chapter 6, we will introduce effective and very effective spectra.

1 The Nisnevich topology

Nisnevich localization is relatively standard. This is obtained by defining the Nisnevich topology on Sm_S, and then imposing the usual sheaf condition. Consequently, the Nisnevich localization functor is a standard sheafification functor, and automatically enjoys the nice formal properties of sheafification. For example, it is an exact functor.

Definition 1.1. Let $X \in \mathrm{Sm}_S$. A Nisnevich cover of X is a finite family of étale morphisms $\{p_i : U_i \to X\}_{i \in I}$ such that there is a filtration

$$\emptyset \subseteq Z_n \subseteq Z_{n-1} \subseteq \cdots \subseteq Z_1 \subseteq Z_0 = X$$

of X by finitely presented closed subschemes such that for each strata $Z_m \setminus Z_{m+1}$, there is some p_i such that

$$p_i^{-1}(Z_m \setminus Z_{m+1}) \to Z_m \setminus Z_{m+1}$$

admits a section.

Example 1.2. Any Zariski cover is a Nisnevich cover. Any Nisnevich cover is an étale cover.

Example 1.3. Let k be a field of characteristic not 2, $S = \operatorname{Spec} k$ and $a \in k^\times$. Consider the covering

$$\begin{array}{ccc} & & \mathbb{A}^1 \setminus \{0\} \\ & & \big\downarrow{\scriptstyle x^2} \\ \mathbb{A}^1 \setminus \{a\} & \hookrightarrow & \mathbb{A}^1 \end{array}.$$

This forms a Nisnevich cover with the filtration $\emptyset \subseteq \{a\} \subseteq \mathbb{A}^1$ iff $\sqrt{a} \in k$.

It turns out to check that something is a Nisnevich sheaf, it suffices to check it for very particular covers with two opens.

Definition 1.4. An elementary distinguished square is a pullback diagram

$$\begin{array}{ccc} U \times_X V & \longrightarrow & V \\ \big\downarrow & & \big\downarrow{\scriptstyle p} \\ U & \xrightarrow{\ i\ } & X \end{array}$$

of S-schemes in Sm_S such that i is a Zariski open immersion, p is étale, and $p^{-1}(X \setminus U) \to X \setminus U$ is an isomorphism of schemes, where $X \setminus U$ is equipped with the reduced induced scheme structure.

$\{U, V\}$ forms a Nisnevich cover of X with filtration $\emptyset \subseteq X \setminus U \subseteq X$.

Definition 1.5. We define $\mathrm{Spc}_S = L_{\mathrm{Nis}}\mathcal{P}(\mathrm{Sm}_S)$ to be the full subcategory of $\mathcal{P}(\mathrm{Sm}_S)$ consisting of presheaves that satisfy descent with respect to Nisnevich covers. Such presheaves are also said to be Nisnevich local. This is an accessible subcategory of $\mathcal{P}(\mathrm{Sm}_S)$ and admits a localization functor $L_{\mathrm{Nis}} : \mathcal{P}(\mathrm{Sm}_S) \to \mathrm{Spc}_S$.

Example 1.6. Every representable functor satisfies Nisnevich descent, since they in fact satisfy étale descent. Note that these functors are valued in discrete spaces.

Lemma 1.7. *Then $F \in \mathcal{P}(\mathrm{Sm}_S)$ is Nisnevich local iff $F(\emptyset) \simeq *$ and for every elementary distinguished square*

$$\begin{array}{ccc} U \times_X V & \longrightarrow & V \\ \downarrow & & \downarrow p \\ U & \xrightarrow{\ i\ } & X \end{array}$$

the induced diagram

$$\begin{array}{ccc} F(X) & \longrightarrow & F(V) \\ \downarrow & & \downarrow \\ F(U) & \longrightarrow & F(U \times_X V) \end{array}$$

is a pullback diagram.

Note that the "only if" direction is immediate from definition, and doesn't require the assumption on S.

Corollary 1.8. *If*

$$\begin{array}{ccc} U \times_X V & \longrightarrow & V \\ \downarrow & & \downarrow p \\ U & \xrightarrow{\ i\ } & X \end{array}$$

is an elementary distinguished square, then, when considered a square Spc_S, this is a pushout diagram.

In particular, this holds when $\{U, V\}$ is a Zariski cover.

2 $\mathbb{A}^1$-LOCALIZATION

Definition 2.1. A presheaf $F \in \mathcal{P}(\mathrm{Sm}_S)$ is $\mathbb{A}^1$-local if the natural map $F(X \times \mathbb{A}^1) \to F(X \times \{0\})$ is an equivalence for all X. We write $L_{\mathbb{A}^1}\mathcal{P}(\mathrm{Sm}_S)$ for the full subcategory of $\mathbb{A}^1$-local presheaves, and $L_{\mathbb{A}^1 \wedge \mathrm{Nis}}\mathcal{P}(\mathrm{Sm}_S) = \mathrm{Spc}_S^{\mathbb{A}^1}$ for the presheaves that are *both* Nisnevich local and $\mathbb{A}^1$-local.

Therefore we get a square of accessible localizations

$$\begin{array}{ccc} \mathcal{P}(\mathrm{Sm}_S) & \xrightarrow{L_{\mathrm{Nis}}} & L_{\mathrm{Nis}}\mathcal{P}(\mathrm{Sm}_S) \equiv \mathrm{Spc}_S \\ \downarrow \widetilde{L_{\mathbb{A}^1}} & & \downarrow L_{\mathbb{A}^1} \\ L_{\mathbb{A}^1}\mathcal{P}(\mathrm{Sm}_S) & \xrightarrow{\widetilde{L_{\mathrm{Nis}}}} & L_{A^1 \wedge \mathrm{Nis}}\mathcal{P}(\mathrm{Sm}_S) \equiv \mathrm{Spc}_S^{\mathbb{A}^1} \equiv \mathcal{H}(S) \end{array}$$

We also write the composite $\mathcal{P}(\mathrm{Sm}_S) \to \mathrm{Spc}_S^{\mathbb{A}^1}$ as L_{Mot}.

Remark 2.2. Note that if we think of each of these as subcategories, $L_{\mathbb{A}^1}$ and $\widetilde{L_{\mathbb{A}^1}}$ are not the same functors.

Remark 2.3. A representable functor is usually not $\mathbb{A}^1$-local. Hence if $X \in \mathrm{Sm}_S$, the resulting sheaf $L_{\mathrm{Mot}}X \in \mathrm{Spc}_S^{\mathbb{A}^1}$ is usually not discrete. If X is already $\mathbb{A}^1$-local, then we say X is $\mathbb{A}^1$-*rigid*. For example, $\mathbb{G}_m$ is $\mathbb{A}^1$-rigid.

Unlike L_{Nis}, the $\mathbb{A}^1$-localization functor $L_{\mathbb{A}^1}$ is not a sheafification functor. Thus, *a priori*, the only nice property of it we know is that it is a left adjoint. To remedy for this, we describe an explicit construction of $\widetilde{L_{\mathbb{A}^1}}$, and then observe that

Lemma 2.4. $L_{\mathrm{Mot}} \simeq (L_{\mathrm{Nis}}\widetilde{L_{\mathbb{A}^1}})^{\omega}$.

The functor $\widetilde{L_{\mathbb{A}^1}}$ is better known as $\mathrm{Sing}^{\mathbb{A}^1}$.

Definition 2.5. Define the "affine n-simplex" Δ^n by

$$\Delta^n = \mathrm{Spec}\, k[x_0, \ldots, x_n]/(x_0 + \cdots + x_n = 1).$$

This forms a cosimplicial scheme in the usual way.

For $X \in \mathcal{P}(\mathrm{Sm}_S)$, we define $\mathrm{Sing}^{\mathbb{A}^1} X \in \mathcal{P}(\mathrm{Sm}_S)$ by

$$(\mathrm{Sing}^{\mathbb{A}^1} X)(U) = |X(U \times \Delta^{\bullet})|.$$

It is then straightforward to check that

Lemma 2.6. $\widetilde{L_{\mathbb{A}^1}} \simeq \mathrm{Sing}^{\mathbb{A}^1}$.

Corollary 2.7. $\widetilde{L_{\mathbb{A}^1}}$ *preserves finite products. Hence so does* L_{Mot}.

Example 2.8. Let $E \to X$ be a (Nisnevich-)locally trivial $\mathbb{A}^n$-bundle. Then $E \to X$ is an $\mathbb{A}^1$-equivalence in $L_{\mathrm{Nis}}\mathcal{P}(\mathrm{Sm}_S)$.

Example 2.9. We claim that $\mathbb{P}^1 \cong \Sigma\mathbb{G}_m = S^1 \wedge \mathbb{G}_m \in \mathrm{Spc}_S^{\mathbb{A}^1}$. Here we are working with pointed objects so that suspensions make sense.

Indeed, we have a Zariski open cover of $\mathbb{P}^1$ given by

$$\begin{array}{ccc} \mathbb{A}^1 \setminus \{0\} & \longrightarrow & \mathbb{A}^1 \\ \downarrow & & \downarrow{\scriptstyle x^{-1}} \\ \mathbb{A}^1 & \xrightarrow{\;x\;} & \mathbb{P}^1 \end{array}$$

Since this is in particular an elementary distinguished square, it is also a pushout in $(\mathrm{Spc}_S)_*$. Now apply $L_{\mathbb{A}^1}$ to this diagram, which preserves pushouts since it is a left adjoint. Since $L_{\mathbb{A}^1}\mathbb{A}^1 \simeq *$ the claim follows.

If we "replace" only one of the $\mathbb{A}^1$'s with $*$, we see that this also says

$$\Sigma\mathbb{G}_m = \mathbb{A}^1/(\mathbb{A}^1 \setminus \{0\}).$$

In general, we have

Lemma 2.10.

$$\mathbb{A}^n \setminus \{0\} \cong S^{n-1} \wedge \mathbb{G}_m^n$$
$$\mathbb{P}^n/\mathbb{P}^{n-1} \cong S^n \wedge \mathbb{G}_m^n.$$

It is convenient to introduce the following notation:

Definition 2.11. We define $S^{p,q} = \mathbb{G}_m^{\wedge q} \wedge (S^1)^{\wedge(p-q)}$ whenever it makes sense.

3 Homotopy sheaves

Naively, one would like to define the motivic homotopy groups of $X \in (\mathrm{Spc}_S^{\mathbb{A}^1})_*$ as $[S^{p,q}, X]_*$. This is an abelian group. We can do better than that, and produce $\pi_{p,q}X$ as a *sheaf*.

Definition 3.1. Let $X \in \mathrm{Spc}_S$. We define $\pi_0^{\mathrm{Nis}}(X)$ to be the Nisnevich sheafification of the presheaf of sets

$$U \mapsto [U, X]_{\mathrm{Spc}_S}.$$

If $X \in (\mathrm{Spc}_S)_*$ and $n \geq 1$, define $\pi_n^{\mathrm{Nis}}(X)$ to be the Nisnevich sheafification of the presheaf of groups

$$U \mapsto [S^n \wedge U_+, X]_{(\mathrm{Spc}_S)_*}.$$

In general, if $X \in \mathcal{P}(\mathrm{Sm}_S)$, we define $\pi_n^{\mathrm{Nis}}(X) = \pi_n^{\mathrm{Nis}}(L_{\mathrm{Nis}}X)$. Finally, we define

$$\pi_n^{\mathbb{A}^1}(X) = \pi_n^{\mathrm{Nis}}(L_{\mathrm{Mot}}X).$$

Corollary 3.2. *If $F \to X \to Y$ is a fiber sequence in $(\mathrm{Spc}_S^{\mathbb{A}^1})_*$, then there is a long exact sequence*

$$\cdots \pi_{n+1}^{\mathbb{A}^1} Y \to \pi_n^{\mathbb{A}^1} F \to \pi_n^{\mathbb{A}^1} X \to \pi_n^{\mathbb{A}^1} Y \to \cdots$$

of Nisnevich sheaves.

Proof. The forgetful functor $\mathrm{Spc}_S^{\mathbb{A}^1} \to \mathcal{P}(\mathrm{Sm}_S)$ is a right adjoint, hence preserves fiber sequences. So fiber sequences in $\mathrm{Spc}_S^{\mathbb{A}^1}$ are computed objectwise. Then note that sheafification is exact. □

Here it is essential that we did not include $L_{\mathbb{A}^1}$ in the definition of $\pi_n^{\mathbb{A}^1}(X)$, since $L_{\mathbb{A}^1}$ is not exact.

Definition 3.3. If $X \in \mathcal{P}(\mathrm{Sm}_S)$, we say X is $\mathbb{A}^1$-connected if the canonical map $X \to S$ induces an isomorphism of sheaves $\pi_0^{\mathbb{A}^1} X \to \pi_0^{\mathbb{A}^1} S = *$.

Given $X \in \mathrm{Spc}_S$, to check if X is $\mathbb{A}^1$-connected, it turns out it suffices to check that $\pi_0^{\mathrm{Nis}}(X)$ is trivial.

Proposition 3.4 (Unstable $\mathbb{A}^1$-connectivity)**.** *Let $X \in \mathcal{P}(\mathrm{Sm}_S)$. Then the canonical map*

$$X \to L_{\mathrm{Mot}} X$$

induces an epimorphism

$$\pi_0^{\mathrm{Nis}} X \to \pi_0^{\mathrm{Nis}} L_{\mathrm{Mot}} X = \pi_0^{\mathbb{A}^1} X.$$

Proof. Since $\pi_0^{\mathrm{Nis}} X \to \pi_0^{\mathrm{Nis}} L_{\mathrm{Nis}} X$ is an isomorphism, it suffices to show that $\pi_0^{\mathrm{Nis}} X \to \pi_0^{\mathrm{Nis}} \mathrm{Sing}^{\mathbb{A}^1} X(U)$ is always an epimorphism. This follows by inspection. □

The final property of π_n^{Nis} we note is that over a perfect field, π_n^{Nis} is *unramified.* Roughly speaking, it says

$$\pi_n^{\mathrm{Nis}}(X)(U) \to \pi_n^{\mathrm{Nis}}(X)(\operatorname{Spec} k(U))$$

is injective for any $U \in \mathrm{Sm}_S$. We cannot exactly say this because $\operatorname{Spec} k(U)$ is in general not smooth over S.

4 Thom spaces

Definition 4.1. Let $E \to X$ be a vector bundle. Then we define the *Thom space* to be

$$\mathrm{Th}(E) = E/E^{\times}.$$

Proposition 4.2.

$$\mathrm{Th}(E) \cong \mathbb{P}(E \oplus 1)/\mathbb{P}(E).$$

Theorem 4.3 (Purity theorem)**.** *Let $Z \hookrightarrow X$ be a closed embedding in Sm_S with normal bundle ν_Z. Then we have an equivalence (in $\mathrm{Spc}_S^{\mathbb{A}^1}$).*

$$\frac{X}{X \setminus Z} \to \mathrm{Th}(\nu_Z).$$

Proof idea. The first geometric input is the construction of a bundle of closed embeddings over $\mathbb{A}^1$ whose fiber over $\{0\}$ is (ν_Z, Z) and (X, Z) elsewhere. This has a very explicit description:

$$D_Z X = \mathrm{Bl}_{Z\times_S\{0\}}(X \times_S \mathbb{A}^1) \setminus \mathrm{Bl}_{Z\times_S\{0\}}(X \times_S \{0\}).$$

Indeed, the fiber over $\{0\}$ is $\mathbb{P}(\nu_Z \oplus \mathcal{O}_Z) \setminus \mathbb{P}(\nu_Z)$, which is canonically isomorphic to ν_Z. (This construction is known as "deformation to the normal cone")

The second step shows that in $\mathcal{H}(S)$, we have a homotopy pushout squares

$$\begin{array}{ccccc}
\dfrac{\nu_Z}{\nu_Z \setminus Z} & \longleftarrow & Z & \longrightarrow & \dfrac{X}{X \setminus Z} \\
\downarrow & & \downarrow & & \downarrow \\
\dfrac{D_Z X}{D_Z X \setminus Z \times \mathbb{A}^1} & \longleftarrow & Z \times \mathbb{A}^1 & \longrightarrow & \dfrac{D_Z X}{D_Z X \setminus Z \times \mathbb{A}^1}
\end{array}$$

To prove this, one uses Nisnevich descent to reduce to the affine case. □

5 Stable motivic homotopy theory

The stable motivic homotopy category is obtained by inverting $\mathbb{P}^1$ in $\mathcal{H}(S)_*$:

$$\mathcal{SH}(S) = \mathcal{H}(S)_*[(\mathbb{P}^1)^{-1}].$$

More generally, suppose we have a presentably symmetric monoidal ∞-category $\mathcal{C}$ (i.e. $\mathcal{C} \in \mathrm{CAlg}(\mathcal{P}r^L)$) and $X \in \mathcal{C}$. We can then define

$$\mathrm{Stab}_X(\mathcal{C}) = \mathrm{colim}\left(\mathcal{C} \xrightarrow{-\otimes X} \mathcal{C} \xrightarrow{-\otimes X} \mathcal{C} \xrightarrow{-\otimes X} \cdots \right),$$

or equivalently

$$\mathrm{Stab}_X(\mathcal{C}) = \lim\left(\mathcal{C} \xleftarrow{(-)^X} \mathcal{C} \xleftarrow{(-)^X} \mathcal{C} \xleftarrow{(-)^X} \cdots \right),$$

where these (co)limits are taken in the category of large categories (or equivalently $\mathcal{P}r^L/\mathcal{P}r^R$).

Theorem 5.1 (Robalo)**.** *If X is* symmetric*, i.e. the cyclic permutation on $X \otimes X \otimes X$ is homotopic to the identity, then* $\mathrm{Stab}_X(\mathcal{C}) \in \mathrm{CAlg}(\mathcal{P}r^L)$ *is symmetric monoidal, and the natural map $\mathcal{C} \to \mathrm{Stab}_X(\mathcal{C})$ is universal among maps in* $\mathrm{CAlg}(\mathcal{P}r^L)$ *that send X to an invertible object.*

Note that there is always a map in $\mathrm{CAlg}(\mathcal{P}r^L)$ satisfying this universal property, and $\mathrm{Stab}_X(\mathcal{C})$ always exists. The condition in the theorem ensures these two agree.

It is easy to check that $\mathbb{P}^1$ is indeed symmetric by elementary row and column operations, so we can define

Definition 5.2. $\mathcal{SH}(S) = \mathcal{H}(S)_*[(\mathbb{P}^1)^{-1}]$.

As in the topological world, we have an adjunction

$$\mathcal{H}(S)_* \underset{\Omega^\infty_{\mathbb{P}^1}}{\overset{\Sigma^\infty_{\mathbb{P}^1}}{\rightleftarrows}} \mathcal{SH}(S)$$

Note that $\Sigma^\infty_{\mathbb{P}^1}$ is by definition symmetric monoidal, and the fact that the tensor product preserves colimits in both variables tells us how to compute the tensor product in $\mathcal{SH}(S)$.

Definition 5.3. For $E \in \mathcal{SH}(S)$ and $i, j \in \mathbb{Z}$, we define

$$\pi_{p,q}(E) = \pi_0^{\mathbb{A}^1}(\Omega^\infty_{\mathbb{P}^1}(E \wedge S^{-p,-q})).$$

Of course, these also lead to homotopy *groups* given by the global sections of the homotopy sheaves.

Definition 5.4. For $E \in \mathcal{SH}(S)$ and $X \in (\mathrm{Sm}_S)_*$, and $p, q \in \mathbb{Z}$, we define

$$\begin{aligned} E_{p,q}(X) &= \pi_{p,q}(\Sigma^\infty_{\mathbb{P}^1} X \wedge E) \\ E^{p,q}(X) &= [\Sigma^\infty_{\mathbb{P}^1} X, \Sigma^{p,q} E]_{\mathcal{SH}(S)}. \end{aligned}$$

We shall briefly introduce three examples of motivic spectra.

Example 5.5. Motivic cohomology is represented by a spectrum $H\mathbb{Z}$. If U is smooth, motivic cohomology is equivalent to Bloch's higher Chow groups:

$$H\mathbb{Z}^{p,q}(U_+) \cong CH^q(U, 2q - p).$$

There are multiple ways one can construct $H\mathbb{Z}$, and I shall describe three. These mimic how one constructs the classical $H\mathbb{Z}$. These work over any perfect field, except for the second which only produces the right spectrum when the characteristic is 0.

1. Classically, we have the ∞-category Ch(Ab) of chain complexes of abelian groups, and singular chains defines a natural map $i^* : \mathrm{Sp} \to \mathrm{Ch}(\mathrm{Ab})$. This admits a right adjoint i_*, and we can define $H\mathbb{Z} = i_* i^* \mathbb{S}$.

   In the motivic world, we can define the triangulated category of motives $\mathrm{DM}(k)$, which receives a map $\mathcal{SH}(S) \to \mathrm{DM}(k)$ and we can repeat the previous paragraph.

2. Classically, we can construct $H\mathbb{Z}$ as an infinite loop space directly using the Eilenberg–Maclane spaces, which one can in turn construct using the Dold–Thom theorem as $\mathrm{Sym}^\infty(S^n)$. This works motivically as well.

3. Finally, we can define $H\mathbb{Z} = \tau_{\leq 0}\mathbb{S}$. Motivically, we can define $H\mathbb{Z}$ as the zero slice of $\mathbb{S}$, which we will discuss later.

Example 5.6. We can construct a motivic spectrum KGL that represents (homotopy) K-theory (which agrees with algebraic K-theory for sufficiently nice schemes). This is obtained by constructing BGL (as the colimit of Grassmannians), and then showing that $\Omega_{\mathbb{P}^1}(BGL \times \mathbb{Z}) \cong BGL \times \mathbb{Z}$. We then have an explicit infinite loop space.

Example 5.7. We can construct the algebraic cobordism spectrum MGL in the same way as we produce MO or MU. There is a universal vector bundle $\gamma_n \to BGL_n$. We then define

$$MGL = \operatorname{colim} \Sigma^{-2n,-n} \mathrm{Th}(\gamma_n).$$

6 Effective and very effective motivic spectra

The category of (non-motivic) spectra admits a t-structure, where the connective objects $\mathrm{Sp}_{\geq 0}$ is the subcategory generated by S^n for $n \geq 0$ under (sifted) colimits. In the motivic world, we have two kinds of spheres — S^1 and $\mathbb{G}_m$, which makes everything bigraded. Thus, we can have different notions of connectivity depending of which spheres we use.

Definition 6.1. Let $\mathcal{SH}^{\mathrm{eff}}(S)$ be the smallest stable subcategory (closed under direct sums) of $\mathcal{SH}(S)$ containing all $\Sigma^\infty_{\mathbb{P}^1} X_+$ for $X \in \mathrm{Sm}_S$ and closed under colimits. This is the category of *effective* spectra.

The inclusion $\mathbb{G}_m^n \wedge \mathcal{SH}^{\mathrm{eff}}(S) \hookrightarrow \mathcal{SH}(S)$ admits a right adjoint f_n. This defines the *slice filtration*

$$\cdots \to f_{n+1}E \to f_n E \to f_{n-1} E \to \cdots .$$

Definition 6.2. The n-slice of E, denoted $s_n E$, is defined by the cofiber sequence

$$f_{n+1}E \to f_n E \to s_n E.$$

On the other hand, we can only allow for non-negative powers of S^i but include negative powers of $\mathbb{G}_m$. This in fact defines a t-structure on $\mathcal{SH}(S)$.

Definition 6.3. Let $\mathcal{SH}(S)_{\geq 0}$ be the subcategory of $\mathcal{SH}(S)$ generated by $\Sigma^\infty_{\mathbb{P}^1} X_+ \wedge \mathbb{G}_m^{-q}$ under extensions and colimits.

Theorem 6.4. *This forms part of a t-structure on $\mathcal{SH}(S)$, called the* homotopy t-structure*.*

In the case of a perfect field, but not in general, we can characterize the homotopy t-structure by the homotopy sheaves.

Theorem 6.5. *If $S = \operatorname{Spec} k$ and k is a perfect field, then*

$$\begin{aligned}\mathcal{SH}(S)_{\geq 0} &= \{E \in \mathcal{SH}(S) \mid \pi_{p,q}(E) = 0 \text{ whenever } p - q < 0\}\\ \mathcal{SH}(S)_{\leq 0} &= \{E \in \mathcal{SH}(S) \mid \pi_{p,q}(E) = 0 \text{ whenever } p - q > 0\}\end{aligned}$$

Finally, we can intersect these two.

Definition 6.6. We define

$$\mathcal{SH}^{\text{veff}}(S) = \mathcal{SH}^{\text{eff}}(S) \cap \mathcal{SH}(S)_{\geq 0}.$$

Equivalently, it is the full subcategory of $\mathcal{SH}(S)$ that generated by $\Sigma^\infty_{\mathbb{P}^1} X_+$ and under colimits. This is the category of *very effective spectra.*

Since the smash product preserves colimits and $X_+ \wedge Y_+ = (X \times)_+$, we see that

Proposition 6.7. *$\mathcal{SH}(S)^{\text{veff}}$ is closed under the smash product.*

Example 6.8. MGL is very effective. To show this, we have to show that for $\gamma_n \to BGL_n$ the universal vector bundle, the Thom spectrum $\Sigma^{-2n,-n}\Sigma^\infty_{\mathbb{P}^1} \operatorname{Th}(\gamma_n)$ is in very effective. This is true for rank n vector bundles in general, since it is true for trivial vector bundles, and vector bundles are locally trivial.