Borwein–Borwein integrals and sums

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The sinc function is defined by

$$\operatorname{sinc}(x) = \frac{\sin(x)}{x}.$$

A standard contour integral tells us

$$\int_{-\infty}^{\infty} \operatorname{sinc}(x)\ \mathrm{d}x = \pi.$$

Alternatively, we can observe that

$$\operatorname{sinc}(x) = \frac{1}{2}\int_{-1}^{1} e^{ikt}\ \mathrm{d}k.$$

So up to some factors $\operatorname{sinc} x$ is the Fourier transform of the indicator function of $[-1, 1]$. The preceding integral of $\operatorname{sinc} x$ can be thought of as the value of the Fourier transform at 0. Applying the Fourier inversion formula and carefully keeping track of the coefficients gives us the previous calculation.

David and Jonathan Borwein* observed that we also have

$$\int_{-\infty}^{\infty} \operatorname{sinc}(x)\operatorname{sinc}\left(\frac{x}{3}\right)\ \mathrm{d}x = \pi,$$

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$$\int_{-\infty}^{\infty} \operatorname{sinc}(x)\operatorname{sinc}\left(\frac{x}{3}\right)\operatorname{sinc}\left(\frac{x}{5}\right)\operatorname{sinc}\left(\frac{x}{7}\right)\ \mathrm{d}x = \pi.$$

* David Borwein being the father of Jonathan Borwein

This pattern holds up until

$$\int_{-\infty}^{\infty} \operatorname{sinc}(x) \operatorname{sinc}\left(\frac{x}{3}\right) \cdots \operatorname{sinc}\left(\frac{x}{13}\right) \, \mathrm{d}x = \pi.$$

Afterwards, we have

$$\int_{-\infty}^{\infty} \operatorname{sinc}(x) \operatorname{sinc}\left(\frac{x}{3}\right) \cdots \operatorname{sinc}\left(\frac{x}{15}\right) \, \mathrm{d}x = \frac{467807924713440738696537864469}{467807924720320453655260875000}\pi.$$

As we keep going on, the value continues to decrease.

1 Direct calculation

Turns out it is possible to calculate the integrals above by pure brute force, and this gives explicit formulas for the integrals as we see above.

In general, let $a_0, a_1, a_2, \ldots, a_n$ be a sequence of positive real numbers, and consider the integral

$$\int_{-\infty}^{\infty} \prod_{k=0}^{n} \operatorname{sinc}(a_k x) \, \mathrm{d}x.$$

Let us put aside the $\frac{1}{x}$ factors for a moment, and expand out $\prod_{k=0}^{n} \sin(a_k x)$:

$$\begin{aligned} \prod_{k=0}^{n} \sin(a_k x) &= \frac{1}{(2i)^{n+1}} (e^{ia_0 x} - e^{-ia_0 x}) \prod_{k=1}^{n} (e^{ia_k x} - e^{-ia_k x}) \\ &= \frac{1}{(2i)^{n+1}} \sum_{\gamma \in \{-1,1\}^n} \varepsilon_\gamma (e^{ib_\gamma x} - (-1)^n e^{-ib_\gamma x}), \end{aligned}$$

where

$$b_\gamma = a_0 + \sum_{k=1}^{n} \gamma_k a_k, \quad \varepsilon_\gamma = \prod_{k=1}^{n} \gamma_k.$$

Note that each of the terms in the right-hand sum is some sort of trigonometric function, depending on the value of n mod 2.

The original integral was

$$\int_{-\infty}^{\infty} x^{-n-1} \prod_{k=0}^{n} \sin(a_k x) \, \mathrm{d}x.$$

Since $\sin(a_k x)$ has a simple zero at $x = 0$, we know from this expression that we can integrate this by parts n times and have vanishing boundary terms:

$$\int_{-\infty}^{\infty} x^{-n-1} \prod_{k=0}^{n} \sin(a_k x) \, \mathrm{d}x = \frac{1}{n!} \int_{-\infty}^{\infty} \frac{1}{x} \frac{\mathrm{d}^n}{\mathrm{d}x^n} \prod_{k=0}^{n} \sin(a_k x) \, \mathrm{d}x.$$

We now use the expression above to compute this n-fold derivative, and get

$$\begin{aligned}\int_{-\infty}^{\infty} \prod_{k=0}^{n} \frac{\sin(a_k x)}{x} \, \mathrm{d}x &= \frac{1}{n!} \int_{-\infty}^{\infty} \frac{1}{x} \frac{1}{2^n} \sum_{\gamma \in \{-1,1\}^n} \varepsilon_\gamma b_\gamma^n \sin(b_\gamma x) \, \mathrm{d}x \\ &= \frac{\pi}{2^n n!} \sum_{\gamma \in \{-1,1\}^n} \varepsilon_\gamma b_\gamma^n \, \mathrm{sign}(b_\gamma).\end{aligned}$$

We claim this is equal to $\frac{\pi}{a_0}$ when $a_0 \geq \sum_{k=1}^{n} a_k$, and smaller otherwise. Indeed, this is exactly the condition that all the b_γ are positive, so that the sign term disappears. The remaining claim is then that

$$\sum_{\gamma \in \{-1,1\}} \varepsilon_\gamma b_\gamma^n = 2^n n! \prod_{k=1}^{n} a_k.$$

Indeed, this follows by considering the n^{th} Taylor coefficient of the equality

$$e^{a_0 t} \prod_{k=1}^{n} (e^{a_k t} - e^{-a_k t}) = \sum_{\gamma \in \{-1,1\}^n} \varepsilon_\gamma e^{b_\gamma t},$$

where on the left we use that $e^{a_k t} - e^{-a_k t} = 2a_k t$.

2 Fourier transform perspective

The preceeding calculation was not very enlightening, but at least it gives precise numbers. There is a more enlightening approach, beginning with our previous observation that sinc is the Fourier transform of the indicator function of $[-1, 1]$, up to some factors.

To get this going, let us first get our conventions straight. We define our Fourier transforms by

$$\mathcal{F}\{f\}(k) = \hat{f}(k) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i k x} \, \mathrm{d}x$$

Then for any function f, we have

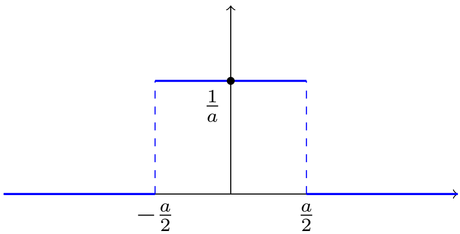
$$\int_{-\infty}^{\infty} f(x) \, \mathrm{d}x = \tilde{f}(0).$$

Why is this useful? In general, it is difficult to say anything about the integral of products. However, the Fourier transform of a product is the convolution of the Fourier transforms, which is an operation we understand pretty well.

With our convention, we have

$$\mathcal{F}\{\operatorname{sinc} ax\}(k) = \begin{cases} \frac{\pi}{a} & |k| < \frac{a}{2\pi} \\ 0 & \text{otherwise} \end{cases} \equiv \chi_{a/\pi}(k).$$

Here for any $a > 0$, the function $\chi_a(x)$ is given by



Note that the area under χ_a is always 1. Fourier transforms take products to convolutions, and convolving with χ_a is pretty simple:

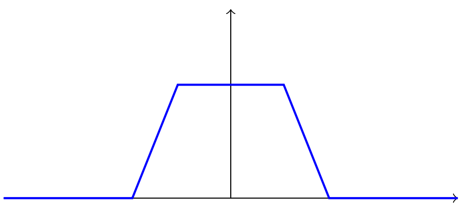
$$(\chi_a * f)(x) = \frac{1}{a}\int_{x-a/2}^{x+a/2} f(u)\,\mathrm{d}u.$$

In words, the value of $\chi_a * f$ at x is the average of the values of f in $[x-a/2, x+a/2]$.

With this in mind, we can look at

$$\int_{-\infty}^{\infty} \prod_{k=0}^{n} \operatorname{sinc}(a_k x)\,\mathrm{d}x = (\chi_{a_0/\pi} * \chi_{a_1/\pi} * \cdots * \chi_{a_n/\pi})(0).$$

We start with the function χ_{a_0}, which is depicted above. Convolving with χ_{a_1} gives a piecewise linear function



Crucially, when $a_1 \leq a_0$, the value at 0 is unchanged, since the function is constant on $\frac{1}{\pi}[-a_0, a_0] \supseteq \frac{1}{\pi}[-a_1, a_1]$. The resulting function is constantly $\frac{\pi}{a_0}$ on the interval $\frac{1}{\pi}[-(a_0 - a_1), a_0 - a_1]$.

When we further convolve with $\chi_{a_2/\pi}$, if $a_1 + a_2 \leq a_0$, the resulting function is constantly $\frac{\pi}{a_0}$ on the interval $\frac{1}{\pi}[-(a_0 - a_1 - a_2), a_0 - a_1 - a_2]$. In general, this tells us that as long as $a_0 \geq a_1 + \cdots + a_n$, the integral will still be $\frac{\pi}{a_0}$, and gets smaller afterwards.

3 Sums

Let us move on to the series version. We also claim that

$$\sum_{n\in\mathbb{Z}} \operatorname{sinc}(n) = \pi.$$

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This continues to hold until $\operatorname{sinc}(\frac{x}{13})$ but fails when we include the $\operatorname{sinc}\left(\frac{x}{15}\right)$ term. Coincidence? We might hope, naïvely, that the correct result is

$$\sum_{n\in\mathbb{Z}} \prod_{k=0}^{N} \operatorname{sinc}\left(\frac{n}{2k+1}\right) = \int_{-\infty}^{\infty} \prod_{k=0}^{N} \operatorname{sinc}\left(\frac{x}{2k+1}\right)\, \mathrm{d}x.$$

This is in fact true, for $N \leq 40248$. Number theorists will be delighted to learn that this follows from the Poisson summation formula.

Theorem 3.1 (Poisson summation formula)**.** *Let $f : \mathbb{R} \to \mathbb{R}$ be compactly supported, piecewise continuous and continuous at integer points. Then*

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \hat{f}(n).$$

The previous observation follows from taking $f(x) = \mathcal{F}\left\{\prod_{k=0}^{N} \operatorname{sinc}\left(\frac{x}{2k+1}\right)\right\}$, which satisfies the hypothesis of the theorem (it is in fact continuous for $n > 0$). The Fourier inversion theorem then tells us $\hat{f}(-x) = \prod_{k=0}^{N} \operatorname{sinc}\left(\frac{x}{2k+1}\right)$. So the right-hand side is the sum in question, and $f(0)$ is the Borwein integral. Our previous analysis shows that the support of $\hat{f}$ is $\frac{1}{2\pi}[-(a_0 + \cdots + a_n), (a_0 + \cdots + a_n)]$. So $\hat{f}$ vanishes at non-negative integers whenever $\sum \frac{1}{2k+1} < 2\pi$.

Note. It is common for the theorem to be stated for Schwarz functions instead. However, our function is not smooth, but the same proof goes through under our hypothesis.

Corollary 3.2.

$$\sum_{n\in\mathbb{Z}} \prod_{k=0}^{N} \operatorname{sinc} a_k n = \int_{-\infty}^{\infty} \prod_{k=0}^{N} \operatorname{sinc} a_k x \, \mathrm{d}x.$$

if $\sum a_k < 2\pi$.

Proof of theorem. Set

$$g(x) = \sum_{n \in \mathbb{Z}} f(x+n).$$

Then, $g(0) = \sum_{n \in \mathbb{Z}} f(n)$. Note that the sum converges since g is compactly supported, and is continuous at 0 since f is continuous at integer points. Of course, it is also piecewise continuous, since in each open neighbourhood, the sum is finite. So we know the Fourier series of g converges at 0. Recall that the Fourier series is

$$g(x) = \sum_{k \in \mathbb{Z}} \hat{g}_k e^{2\pi i k x},$$

where

$$\begin{aligned}
\hat{g}_k &= \int_{\mathbb{R}/\mathbb{Z}} e^{-2\pi i k x} g(x)\, \mathrm{d}x \\
&= \sum_{a \in \mathbb{Z}^n} \int_{[0,1]} e^{-2\pi i k x} f(x+a)\, \mathrm{d}x = \int_{\mathbb{R}} e^{-2\pi i k x} f(x)\, \mathrm{d}x = \hat{f}(k).
\end{aligned}$$

So

$$\sum_{n \in \mathbb{Z}} f(n) = g(0) = \sum_{k \in \mathbb{Z}} \hat{g}_k = \sum_{k \in \mathbb{Z}} \hat{f}(k). \qquad \square$$